ARMA ORDER DETERMINATION

BY

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CHAPTER I

INTRODUCTION

The autoregressive moving average (ARMA(p,q)) model is the most popular model for forecasting the future behavior of time series data.

Various methods have been developed to determine the best ARMA(p,q) model to fit different kinds of time series data. The purpose of this thesis is to discuss the popular methods used for ARMA(p,q) modeling. Our task will be to determine p and q where p is the autoregressive (AR) order and q is the moving average (MA) order, this process is called order determination.

An introduction to AR(p),MA(q), and ARMA(p,q) models, their autocorrelation function (ACF), and their impulse response (IR) will be presented in chapter two. Also, the key equation for all methods we are going to investigate will be stated.

Chapter three introduces the generalized partial autocorrelation function (GPAC), which will be used to determine the ARMA process order (p and q). Chapter four will discuss the R and S arrays, the S arrays in particular. Theorems concerning their use in order determination will be stated, the shifted S and R arrays will be presented and their relationship to the GPAC will discussed.

In chapter five we will introduce what we think is a general form of the GPAC and S and R arrays by introducing a variation which uses complex functions.

Chapter six will deal with the Pade table and the C-table and how they can be used to determine p and q of an ARMA(p,q) process. Chapter seven will discuss the singular value decomposition (SVD) and how it can be used to determine the ARMA(p,q) process order. In chapter eight simulations and comparisons for various methods will be presented. Finally, in chapter nine, conclusions and some suggestions for future studies are presented.

CHAPTER II

PRELIMINARY ARMA MODELING

The time series that will be discussed here are assumed to be realizations of stochastic processes. The output is z_t and the input is e_t. In all the simulations that were done, e_t will be N(0,1).

The mean of z_t is :

$$\mu_z = E\,[\,z_t\,] \tag{1}$$

and the autocorrelation function is :

$$R_z(\tau) = E\,[(z_t-\mu_z)(z_{t+\tau}-\mu_z)] \tag{2}$$

An unbiased estimator of the mean will be used

$$\hat{\mu}_z = \frac{1}{N}\sum_{t=1}^{N} z_t \tag{3}$$

and an estimator of the autocorrelation function

$$\hat{R}z(\tau) = \frac{1}{N-\tau}\sum_{t=1}^{N-\tau} (z_t-\mu)(z_{t+\tau}-\mu) \tag{4}$$

The normalized autocorrelation function will be

$$f_z(\tau) = \frac{R_z(\tau)}{R_z(0)}$$

Since the ARMA(p,q) process is our main concern , it is worth explaining a little bit about the AR(p),MA(q) and ARMA(p,q) processes and the properties of their autocorrelation functions.

(1) <u>Autoregressive process</u> AR(p)

The autoregressive process is described by :

$$z_t = \phi_1 z_{t-1} + \ldots + \phi_p z_{t-p} + e_t \tag{5}$$

Where z_t is our present output and e_t is discrete white noise $N(0,\sigma^2_e)$.

The transfer function of an AR(p) process will be :

$$F(Z) = 1/[1 - \phi_1 Z^{-1} - \ldots - \phi_p Z^{-p}] \tag{6}$$

To find the autocorrelation function of the AR(p) process, multiply (5) by $z_{t-\tau}$ and take the expected value of each term and the result will be :

$$R_z(\tau) = \phi_1 R_z(\tau-1) + \ldots + \phi_p R_z(\tau-p) + \sigma^2_e \quad \tau=0$$

$$R_z(\tau) = \phi_1 R_z(\tau-1) + \ldots + \phi_p R_z(\tau-p) \qquad \tau > 0 \tag{7}$$

The impulse response for AR(p) satisfies the following equation

$$\pi_\tau - \phi_1 \pi_{\tau-1} - \ldots - \phi_p \pi_{\tau-p} = \delta(\tau)$$

where

$$\delta(\tau) = \begin{cases} 1 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases}$$

so

$$\pi_\tau - \phi_1 \pi_{\tau-1} - \ldots - \phi_p \pi_{\tau-p} = 0 \quad \tau > 0$$

The difference between the autocorrelation function and the impulse response is that

$$f_z(\tau) = f_z(-\tau) \qquad \tau > 0 \qquad \text{(even function)}$$

while

$$\pi_{-\tau} = 0 \qquad \tau > 0$$

Example 2.1 :

$$z_t + 1.7119 z_{t-1} + .81 z_{t-2} = e_t$$

The autocorrelation function :

$$R_z(\tau) + 1.7119 R_z(\tau-1) + .81 R_z(\tau-2) = R_{ze}(\tau)$$

$\tau = 0$

$$R_z(0) + 1.7119 R_z(1) + .81 R_z(2) = 1$$

$\tau = 1$

$$R_z(1) + 1.7119 R_z(0) + .81 R_z(1) = 0$$

$\tau = 2$

$$R_z(2) + 1.7119 R_z(1) + .81 R_z(0) = 0$$

solving three equations with three unknowns

$R_z(0) = 27.573$, $R_z(1) = -26.078$, $R_z(2) = 22.310$

and

$$R_z(\tau) + 1.7119 R_z(\tau-1) + .81 R_z(\tau-2) = 0 \qquad \tau > 2$$

The impulse response :

$$\pi_\tau + 1.7119 \pi_{\tau-1} + .81 \pi_{\tau-2} = \delta(\tau)$$

$\tau = 0$

$$\pi_0 = 1$$

$\tau = 1$

$$\pi_1 + 1.7119 \pi_0 = 0$$

$$\pi_1 = -1.7119$$

$$\pi_\tau + 1.7119 \pi_{\tau-1} + .81 \pi_{\tau-2} = 0 \qquad \tau > 1$$

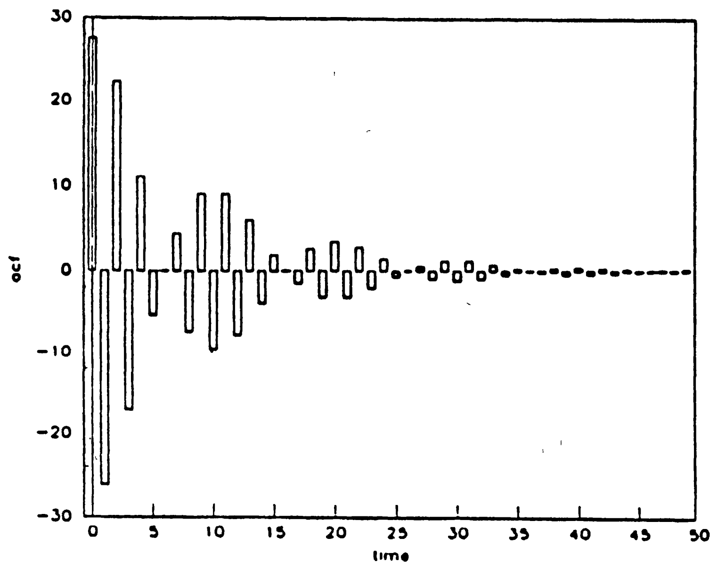

Figure 2.1 The autocorrelation function of example 2.1

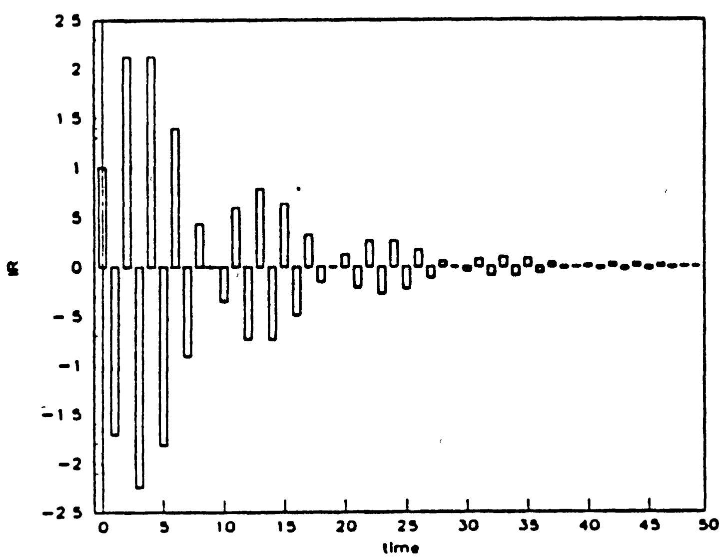

Figure 2.2 The impulse response of example 2.1

(2) The moving average process MA(q)

The moving average process is described by the equation:

$$z_t = e_t - \theta_1 e_{t-1} - \dots - \theta_q e_{t-q} \tag{8}$$

The transfer function will be :

$$F(Z) = 1 - \theta_1 Z^{-1} - \dots - \theta_q Z^{-q} \tag{9}$$

It is not difficult to show that the autocorrelation function of the MA(q) process is :

$$R_z(0) = \sigma^2_e (1 + \theta_1^2 + \dots + \theta_q^2)$$

$$R_z(\tau) = \sigma^2_e (-\theta_\tau + \theta_1\theta_{\tau+1} + \theta_2\theta_{\tau+2} + \dots + \theta_{q-\tau}\theta_q)$$

$$\tau = 1 , 2 , \dots , q$$

$$R_z(\tau) = 0 \qquad \tau > q \tag{10}$$

Example 2.2 :

$z_t = e_t - .5e_{t-1}$

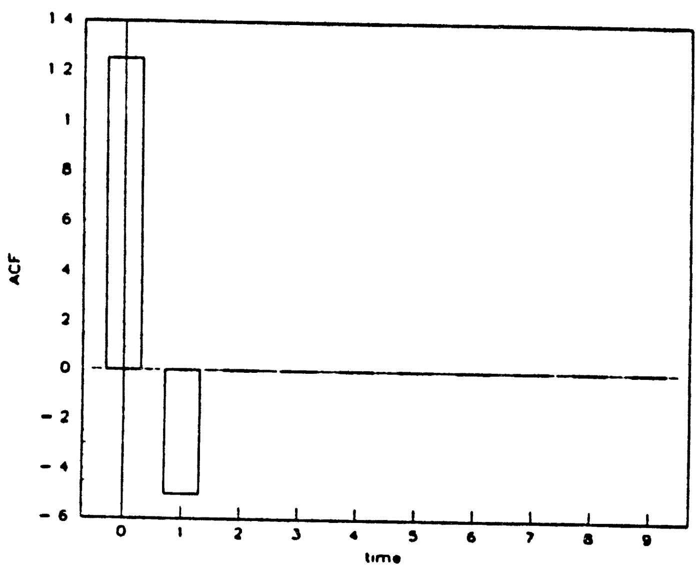

Figure 2.3 The autocorrelation function of example 2.2

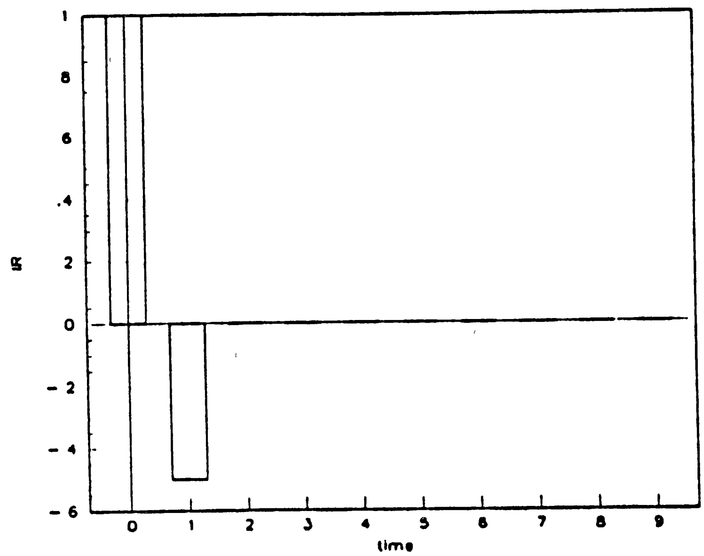

Figure 2.4 The impulse response of example 2.2

(3) <u>The Autoregressive - Moving Average process</u> ARMA(p,q)

This process is a combination of AR(p) and MA(q) . It is described by the equation below :

$$z_t - \phi_1 z_{t-1} - \ldots - \phi_p z_{t-p} = e_t - \theta_1 e_{t-1} - \ldots - \theta_q e_{t-q} \qquad (11)$$

The transfer function for this process will be :

$$F(Z) = \frac{[\, 1 - \theta_1 Z^{-1} - \ldots - \theta_q Z^{-q} \,]}{[\, 1 - \phi_1 Z^{-1} - \ldots - \phi_p Z^{-p} \,]} \qquad (12)$$

and the autocorrelation function is :

$$R_z(\tau) - \phi_1 R_z(\tau-1) - \ldots - \phi_p R_z(\tau-p) = R_{ze}(\tau) - \theta_1 R_{ze}(\tau-1) - \ldots - \theta_q R_{ze}(\tau-q) \qquad (13)$$

Where

$$R_{ze}(\tau) = \begin{cases} \sigma^2_e & \tau = 0 \\ 0 & \tau > 0 \\ \pi_{-\tau}\sigma^2_e & \tau < 0 \end{cases}$$

Where $\pi_{-\tau}$ is the impulse response at time $-\tau$.

When $\tau > q$ equation (13) will reduce to :

$$R_z(\tau) - \phi_1 R_z(\tau-1) - \ldots - \phi_p R_z(\tau-p) = 0 \qquad \tau > q \tag{14a}$$

Dividing the above equation by $R_z(0)$ yields

$$\boxed{f_z(\tau) - \phi_1 f_z(\tau-1) - \ldots - \phi_p f_z(\tau-1) = 0 \qquad \tau > q} \tag{14}$$

This equation is the key to all methods we are going to study .

Example 2.3 :

$$z_t + 1.7119z_{t-1} + .81z_{t-2} = e_t - .5e_{t-1}$$

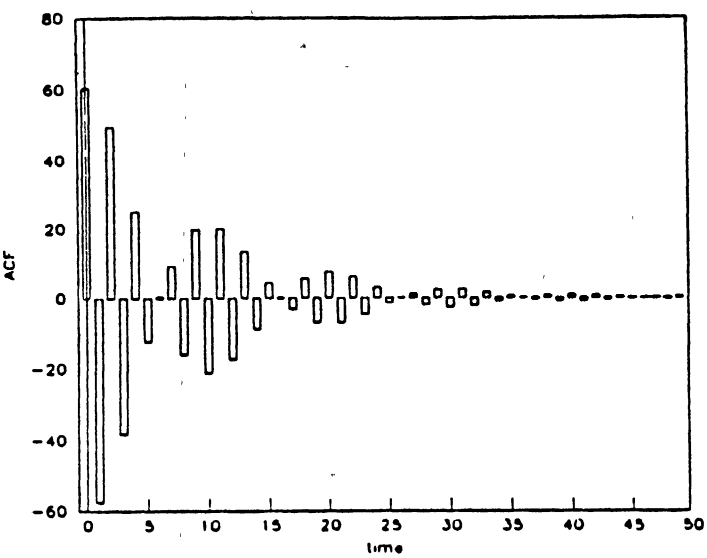

Figure 2.5 The autocorrelation function of example 2.3

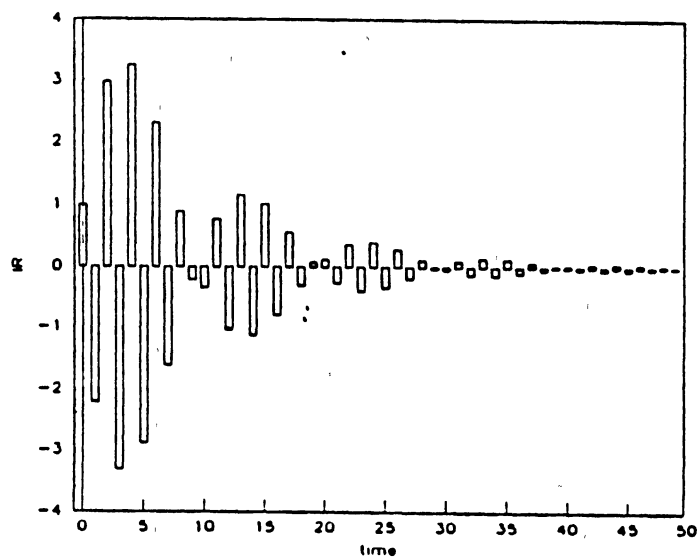

Figure 2.6 The impulse response of example 2.3

CHAPTER III

THE GENERALIZED PARTIAL AUTOCORRELATION FUNCTION

For an MA(q) process the autocorrelation function is enough to determine the order of the system since :

$$f_z(\tau) = 0 \qquad \text{for} \qquad \tau > q \ .$$

On the other hand the autocorrelation alone does not help to decide the order of AR(p), so what is called the partial autocorrelation function (PAC) was introduced .

Define :

$$\begin{bmatrix} f_z(0) & \dots & f_z(k-1) \\ f_z(1) & \dots & f_z(k-2) \\ \vdots & & \vdots \\ f_z(k-1) & \dots & f_z(0) \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kk} \end{bmatrix} = \begin{bmatrix} f_z(1) \\ f_z(2) \\ \vdots \\ f_z(k) \end{bmatrix} \qquad (15)$$

Where (15) is called the Yule - Walker equations. The term ϕ_{kk} is called the partial autocorrelation function (PAC), and it can be found solving (15) using Cramer's rule :

$$\phi_{kk} = \frac{\begin{vmatrix} f_z(0) & \cdots & f_z(k-2) & f_z(1) \\ f_z(1) & \cdots & f_z(k-3) & f_z(2) \\ \vdots & & & \vdots \\ f_z(k-1) & \cdots & f_z(1) & f_z(k) \end{vmatrix}}{\begin{vmatrix} f_z(0) & \cdots & f_z(k-1) \\ \vdots & & \vdots \\ f_z(k-1) & \cdots & f_z(0) \end{vmatrix}} \tag{16}$$

For an AR(p) process :

$$\begin{aligned} \phi_{p1} &= \phi_1 \\ \phi_{p2} &= \phi_2 \\ &\vdots \\ \phi_{pp} &= \phi_p \end{aligned}$$

From equation (14)

$$f_z(\tau) - \phi_1 f_z(\tau-1) - \ldots - \phi_p f_z(\tau-p) = 0 \quad \tau > 0$$

and when $k = p+1$ the last column of the numerator of equation (16) is a linear combination of the other columns, hence $\phi_{kk} = 0$. In fact $\phi_{kk} = 0$ for $k = p+1, p+2, \ldots$. So ϕ_{kk} can be used to determine p for an AR(p) process and the pattern will be as follows

k	1	2		p-1	p	p+1	p+2	P+3	
	ϕ_{11}	ϕ_{22}	...	ϕ_{p-1p-1}	ϕ_p	0	0	0	...

<u>Example 3.1:</u>

$$z_t + 1.7119 z_{t-1} + .81 z_{t-2} = e_t$$

$$\phi 11 = \frac{f_1}{f_0} = -0.9458 \qquad \phi_{22} = \frac{\begin{vmatrix} f_0 & f_1 \\ f_1 & f_2 \end{vmatrix}}{\begin{vmatrix} f_0 & f_1 \\ f_1 & f_0 \end{vmatrix}} = -0.8100$$

$$\phi_{33} = \frac{\begin{vmatrix} f_0 & f_1 & f_1 \\ f_1 & f_0 & f_2 \\ f_2 & f_1 & f_3 \end{vmatrix}}{\begin{vmatrix} f_0 & f_1 & f_2 \\ f_1 & f_0 & f_1 \\ f_2 & f_1 & f_0 \end{vmatrix}} = 0 \qquad \phi_{44} = 0$$

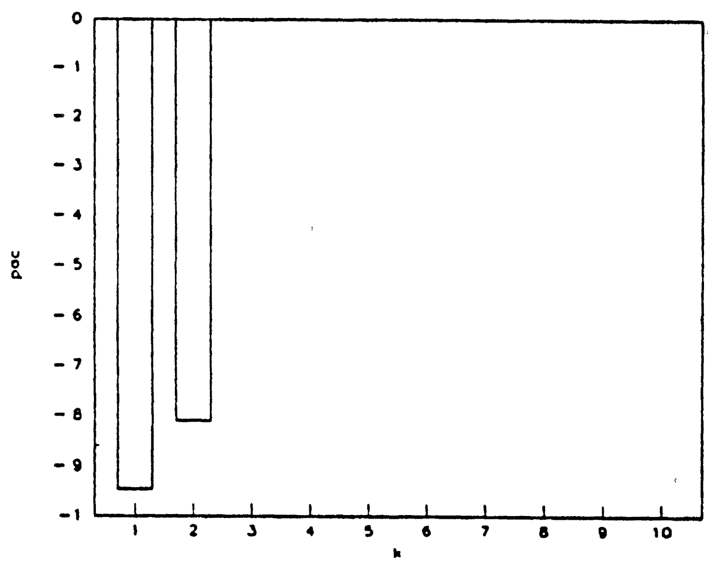

Figure 3.1 ϕ_{kk} for example 3.1

So far we can decide the order of MA(q) using the autocorrelation function (ACF) and the order of AR(p) using the PAC. This raises the question of how to determine the order of a mix of MA(q) and AR(p) which is the ARMA(p,q). Equation (15) can be generalized as :

$$\begin{bmatrix} f_z(j) & \dots & f_z(j-m+1) \\ \vdots & & \\ f_z(j+k-1) & \dots & f_z(j) \end{bmatrix} \begin{bmatrix} \phi^j_{k1} \\ \\ \phi^j_{kk} \end{bmatrix} = \begin{bmatrix} f_z(j+1) \\ \vdots \\ f_z(j+k) \end{bmatrix} \tag{17}$$

The term ϕ^j_{kk} is the generalized partial autocorrelation function (GPAC). Solving for ϕ^j_{kk} :

$$\phi^j_{kk} = \frac{\begin{vmatrix} f_z(j) & \dots & f_z(j-m+2) & f_z(j+1) \\ \vdots & & \vdots & \vdots \\ f_z(j+k-1) & \dots & f_z(j-1) & f_z(j+k) \end{vmatrix}}{\begin{vmatrix} f_z(j) & . & . & . & f_z(j-k+1) \\ \vdots & & & & \vdots \\ f_z(j+k-1) & . & . & . & f_z(j) \end{vmatrix}} \tag{18}$$

At this point one does not have any idea about the p or q of the ARMA(p,q) model, so, we assume k = 1, start varying j, and calculating ϕ^j_{11}, j = 0,1,2,.... Next, assume k = 2 and repeat what we did before, once we reach the right k and j, i.e k = p and j = q

$$\phi^q_{pp} = \frac{\begin{vmatrix} f_z(q) & . & . & . & f_z(q-p+2) & f_z(q+1) \\ \vdots & & & & \vdots & \vdots \\ f_z(q+p-1) & \dots & & & fz(q-1) & f_z(q+p) \end{vmatrix}}{\begin{vmatrix} fz(q) & . & . & . & fz(q-p+1) \\ \vdots & & & & \vdots \\ f_z(q+p-1) & \dots & & & f_z(q) \end{vmatrix}}$$

this is the solution of ϕ_p from equation (14) using Cramer's rule, therefore $\phi^q_{pp} = \phi_p$.

Change k to p+1 and keep j = q

$$
\phi^{q}_{p+1\,p+1} = \frac{\begin{vmatrix} f_z(q) & \dots & f_z(q-p+1) & f_z(q+1) \\ \vdots & & \vdots & \vdots \\ f_z(q+p) & \dots & f_z(q-1) & f_z(q+p+1) \end{vmatrix}}{\begin{vmatrix} f_z(q) & \dots & f_z(q-p) \\ \vdots & & \vdots \\ f_z(q+p) & \dots & f_z(q) \end{vmatrix}}
$$

recalling equation (14)

$$\mathbf{f_z(\tau) - \phi_1 f_z(\tau-1) - \dots - \phi_p f_z(\tau-1) = 0} \qquad \tau > q$$

the last column in the numerator is a linear combination of the others, so the numerator is equal to zero while the denominator is not, hence , $\phi^{q}_{p+1p+1} = 0$. Using the same argument $\phi^{q}_{kk} = 0$ for k = p+1, p+2, ... Also $\phi^{j}_{kk} = (0/0)$ for j = q+1, q+2, ... and k = p+1, p+2, ... because both the numerator and the denominator are zeros.

It is easier to determine the order of the ARMA process if table was formulated as shown below :

TABLE 3.1

THE PATTERN OF THE GPAC

j \ k	1	2	...	p-1	p	p+1	p+2
0	ϕ^{0}_{11}	ϕ^{0}_{22}	...				
1	ϕ^{1}_{11}	ϕ^{1}_{22}	...				
.	.	.					
.	.	.					
.	.	.		ϕ^{q-1}_{p-1p-1}	ϕ^{q-1}_{pp}	ϕ^{q-1}_{p+1p+1}	ϕ^{q-1}_{p+1p+1}
q	ϕ^{q}_{11}	ϕ^{q}_{22}	...	ϕ^{q}_{p-1p-1}	ϕ_{p}	0	0
q+1	ϕ^{q+1}_{11}	ϕ^{q+1}_{22}	...	ϕ^{q+1}_{p+1p+1}	ϕ_{p}	0/0	0/0
q+2	ϕ^{q+2}_{11}	ϕ^{q+2}_{22}	...	ϕ^{q+2}_{p+1p+1}	ϕ_{p}	0/0	0/0

The above table can be used to determine the order of an ARMA process , i.e p and q .

Example 3.2 :

$$z_t + 1.7119 z_{t-1} + .81 z_{t-2} = e_t - .5 e_{t-1}$$

The GPAC of this example is shown in table 3.2.

TABLE 3.2

THE GPAC FOR EXAMPLE 3.2

	1	2	3	4	5
0	-0.9504	-0.8911	-0.3690	-0.1769	-0.0875
1	-0.8596	-0.8100	0.0000	0.0000	0.0000
2	-0.7696	-0.8100	0/0	0/0	0/0
4	-0.4835	-0.8100	0/0	0/0	0/0

CHAPTER IV

THE S AND R ARRAYS

A time series can be described by its correlation function. An equivalent description is the power spectrum of the time series which is the fourier transform of the autocorrelation function. Gray, Morgan, and Houston (1978) suggested a procedure for spectral estimation based upon a numerical integration technique, the so called G_n- and e_n-transformations. It is within the procedure of calculating this spectral density that what is called the S and R arrays appear. It was found by Gray, Kelley, and McIntire (1978) that the S and R arrays can be used for order determination of the ARMA process.

In the following we present the definitions and the theorems necessary to establish the results referred to above.(These results were taken from Gray, Kelley, and McIntire (1978)).

<u>Definition 4.1 :</u>

Let m be an integer, $h > 0$, and f be the normalized autocorrelation function. Also let $f_m = f_z(mh)$

$$H_n[f_m] = \begin{vmatrix} f_m & f_{m+1} & \dots & f_{m+n-1} \\ f_{m+1} & f_{m+2} & \dots & f_{m+n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ f_{m+n-1} & & \dots & f_{m+2n-2} \end{vmatrix} \qquad (19)$$

$H_o[f_m] = 1$; and

$$H_{n+1}[1;f_m] = \begin{vmatrix} 1 & 1 & \dots & 1 \\ f_m & f_{m+1} & \dots & f_{m+n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ f_{m+n-1} & & \dots & f_{m+2n-1} \end{vmatrix} \qquad (20)$$

then

$$R_n(f_m) = \frac{H_n[f_m]}{H_n[1;f_m]} \qquad (21)$$

and

$$S_n(f_m) = \frac{H_n[1;f_m]}{H_n[f_m]} \qquad (22)$$

Pye and Atchison (1973) showed that Rn(fm) and Sn(fm) can be calculated recursively as follows :

$S_o(f_m) = 1$ for $m = 0, \pm 1, \pm 2, \dots$

$R_1(f_m) = f_m$ for $m = 0, \pm 1, \pm 2, \dots$

then

$$R_{n+1}(f_m) = R_n(f_{m+1}) \left[\frac{S_n(f_{m+1})}{S_n(f_m)} - 1 \right] \qquad (23)$$

and

$$S_n(f_m) = S_{n-1}(f_{m+1}) \left[\frac{R_n(f_{m+1})}{R_n((f_m)} - 1 \right] \qquad (24)$$

for $n = 1, 2, \dots$ and $m = 0, \pm 1, \pm 2, \dots$.

Theorem 4.1 :

Let z_t be a stationary ARMA(p,q) process with a normalized autocorrelation function f_m. Suppose that $S_n(f_m)$ and $R_n(f_m)$ are defined, $p > 0$ and $S_n(f_m) \neq 0$.
Then for some integer m_o and some constant $C1 \neq 0$

1) $S_n(f_m) = C1 \quad , \quad m \geq m_o$ (25)

$S_n(f_{m_o-1}) \neq C1$

iff $n = p$ and $m_o = q-p+1$.

2) $$C1 = (-1)^p \left[1 - \sum_{k=1}^{p} \phi_k \right] \qquad (26)$$

Proof :

The constant pattern starts when $n = p$ and $m_o = q-p+1$, so

$$S_p(f_{q-p+1}) = \frac{\begin{vmatrix} 1 & \cdots & 1 & 1 \\ f_{q-p+1} & \cdots & f_q & f_{q+1} \\ f_{q-p+2} & \cdots & f_{q+1} & f_{q+2} \\ \vdots & & \vdots & \vdots \\ f_q & \cdots & f_{q+p-1} & f_{q+p} \end{vmatrix}}{\begin{vmatrix} f_{q-p+1} & \cdots & f_{q-1} & f_q \\ f_{q-p+2} & \cdots & f_q & f_{q+1} \\ \vdots & & \vdots & \vdots \\ f_q & \cdots & f_{q+p-2} & f_{q+p-1} \end{vmatrix}}$$

Using equation (14)

$$f_z(\tau) - \phi_1 f_z(\tau-1) - \ldots - \phi_p f_z(\tau-1) = 0 \qquad \tau > q$$

the last column of the numerator (except the first entry) is a linear combination of the other columns, multiply the first column by ϕ_p, the second by ϕ_{p-1} and so on, then subtract the sum of the results from the last column. Then

$$S_p(f_{q-p+1}) = \frac{\begin{vmatrix} 1 & \cdots & 1 & 1-\sum_{k=1}^{p}\phi_k \\ f_{q-p+1} & \cdots & f_q & 0 \\ \vdots & & \vdots & \vdots \\ f_q & \cdots & f_{q+p-1} & 0 \end{vmatrix}}{\begin{vmatrix} f_{q-p+1} & \cdots & f_{q-1} & f_q \\ f_{q-p+2} & \cdots & f_q & f_{q+1} \\ \vdots & & \vdots & \vdots \\ f_q & \cdots & f_{q+p-2} & f_{q+p-1} \end{vmatrix}}$$

hence,

$$S_p(f_{q-p+1}) = (-1)^p \left[1 - \sum_{k=1}^{p}\phi_k\right] = C_1 .$$

<u>Theorem 4.2 :</u>

Under the conditions of the previous theorem

1) $S_n(f_m) = C_2 \quad , \quad m \le m_1$ (27)

$S_n(f_{m_1+1}) \neq C_2$

iff $n = p$ and $m_1 = -p-q$.

2) $C_2 = -C_1/\phi_p$. (28)

<u>Proof :</u>

The proof can be established in the same manner as the above theorem.

Table 4.1 presents the behavior of the S array for the ARMA(p,q) process, in table 4.2 the sifted S array is used, where the new S array is a shifted version of the old one

$$S_k(j)(NEW) = S_k(j-k+1)(OLD) \tag{29}$$

in all the simulations and examples that will be used the pattern of table 4.2 will be the one that we are looking for.

TABLE 4.1

THE PATTERN OF THE S ARRAY

m \ n	1		p	p+1
$-l$	$S_1(-l)$	...	C_2	*
$-l+1$	$S_1(-l+1)$	...	C_2	*
.	.		.	.
.	.		.	.
.	.		.	.
$-q-p-2$	$S_1(-q-p-2)$	...	C_2	*
$-q-p-1$	$S_1(-q-p-1)$	...	C_2	∞
$-q-p$	$S_1(-q-p)$	...	C_2	
.	.			2q non-
.	.		2q non-	constants
.	.		constants	
$q-p$	$S_1(q-p)$	...		$-C_1$
$q-p+1$	$S_1(q-p+1)$	...	C_1	#
.	.		.	.
.	.		.	.
.	.		.	.
l	$S_1(l)$	...	C_1	#

$* = C_2\ [0/0 - 1]$

$\# = C_1\ [0/0 - 1]$

TABLE 4.2

THE PATTERN OF THE SHIFTED
S ARRAYS

m \ n	1		p	p+1	. . .	p+i
$-l$	$S_1(-l)$	...	C_2	*		U
$-l+1$	$S_1(-l+1)$	...	C_2	*		U
.	.		.	.		.
.	.		.	.		.
.	.		.	.		.
$-q-2$	$S_1(-q-1)$	...	C_2	*		U
$-q-1$	$S_1(-q)$	...	C_2	∞		∞
.	.		.			
.	.		.			
.	.		.			
-1	$S_1(-1)$		2q non-	2q non-		2q non-
0	$S_1(0)$		constants	constants		constants
1	$S_1(1)$					
	.		.			
	.		.			
	.		.			
q	$S_1(q-p+1)$	...	C_1	$-C_1$	...	$(-1)^i C_1$
.	.		.	#		U
.	.		.	.		.
.	.		.	.		.
l	$S_1(l)$	...	C_1	#		U

Equations (23) and (24) can be used easily to verify the ∞ and the $(-1)^i C_1$ patterns.

The following theorem establishes the relationship between the S arrays and the GPAC.

<u>Theorem 4.3 :</u>

The GPAC can be calculated by a ratio of the S arrays as follows :

$$\phi^{j}_{kk} = \frac{-\,S_k(f_{-k+j+1})}{S_k(f_{-k-j})} \tag{30}$$

Proof :

By definition :

$$\frac{S_k(f_{-k+j+1})}{S_k(f_{-k-j})} = \frac{H_{k+1}(1;f_{-k+j+1})\; H_k(f_{-k-j})}{H_k(f_{-k+j+1})\; H_k(1;f_{-k-j})}$$

interchanging columns, it can be seen that

$$H_{k+1}(1;f_{-k-j}) = (-1)^k H_{k+1}(1;f_{-k+j-1})$$

then

$$\frac{S_k(f_{-k+j+1})}{S_k(f_{-k-j})} = \frac{(-1)^k H_k(f_{-k-j})}{H_k(f_{-k+j+1})}$$

using column interchange for both the numerator and the denominator

$$\frac{S_k(f_{-k+j+1})}{S_k(f_{-k-j})} = \frac{-\begin{vmatrix} f_j & f_{j-1} & \cdots & f_{j-k+2} & f_{j+1} \\ f_{j+1} & f_j & \cdots & f_{j-k+3} & f_{j+2} \\ \vdots & & & \vdots & \vdots \\ f_{j+k-1} & f_{j+k-2} & \cdots & f_{j+1} & f_{j+k} \end{vmatrix}}{\begin{vmatrix} f_j & f_{j-1} & \cdots & f_{j-k+1} \\ \vdots & & & \vdots \\ f_{j+k-1} & f_{j+k-2} & \cdots & f_j \end{vmatrix}}$$

the right hand side is $-\phi^{j}_{kk}$. Hence

$$\phi^{j}_{kk} = \frac{-\,S_k(f_{-k+j+1})}{S_k(f_{-k-j})}$$

CHAPTER V

GENERALIZATION OF S & R ARRAYS AND GPAC

M. J. Morton and H. L. Gray (1984) mentioned using $f_m e^{j2\pi wm}$ as an alternative sequence to find the G - spectral estimator. This generalized form will be applied to S & R arrays and also to the GPAC.

(1) Modified S-Array

Based on the above, our previous definitions and theorems will be modified as follows :

Theorem 5.1 :

Let z_t be a stationary ARMA(p,q) process with normalized autocorrelation f_m. Suppose that $S_n(f_m e^{j2\pi wm})$ and $R_n(f_m e^{j2\pi wm})$ are defined. $0 \le w \le .5$, $p > 0$, and $S_n(f_m e^{j2\pi wm}) \neq 0$. Then for some integer m_o and some constant $C_3 \neq 0$

1) $S_n(f_m e^{j2\pi wm}) = C_3 \quad , m \ge m_o$ (31)

$S_n(f_{m_o-1} e^{j2\pi w(m_o-1)}) \neq C_3$

iff $n = p$ and $m_o = q-p+1$. Moreover

2) $C_3 = (-1)^p \left[1 - \sum_{k=1}^{p} e^{j2\pi wk} \phi_k \right]$ (32)

Proof :

The constant pattern starts when $n = p$ and $m_o = q-p+1$, so

$S_p(f_{q-p+1}) =$

$$
\frac{\begin{vmatrix}
1 & \cdots & 1 & 1 \\
e^{j2\pi w(q-p+1)} f_{q-p+1} & \cdots & e^{j2\pi w(q)} f_q & e^{j2\pi w(q+1)} f_{q+1} \\
e^{j2\pi(q-p+2)} f_{q-p+2} & \cdots & e^{j2\pi w(q+1)} f_{q+1} & e^{j2\pi w(q+2)} f_{q+2} \\
\vdots & & \vdots & \vdots \\
e^{j2\pi wq} f_q & \cdots & e^{j2\pi w(q+p-1)} f_{q+p-1} & e^{j2\pi w(q+p)} f_{q+p}
\end{vmatrix}}{\begin{vmatrix}
e^{j2\pi w(q-p+1)} f_{q-p+1} & e^{j2\pi w(q-p+2)} f_{q-p+2} & \cdots & e^{j2\pi wq} f_q \\
\vdots & \vdots & & \vdots \\
e^{j2\pi wq} f_q & e^{j2\pi w(q+1)} f_{q+1} & \cdots & e^{j2\pi(q+p-1)} f_{q+p-1}
\end{vmatrix}}
$$

the numerator can be written as

$$
\begin{vmatrix}
1 & & & & & \\
& e^{j2\pi(q+1)} & & & & \\
& & e^{j2\pi(q+2)} & & & \\
& & & \ddots & & \\
& & & & & e^{j2\pi(q+p)}
\end{vmatrix} *
$$

$$
\begin{vmatrix}
1 & & 1 & 1 \\
e^{-j2\pi wp} f_{q-p+1} & \cdots & e^{-j2\pi w} f_q & f_{q+1} \\
\vdots & & & \vdots \\
e^{-j2\pi w} f_q & \cdots & e^{-j2\pi w} f_{q+p} & f_{q+p+1}
\end{vmatrix}
$$

the denoninator can be written as

$$
\begin{vmatrix}
e^{j2\pi w(q+1)} & & & \\
& e^{j2\pi w(q+2)} & & \\
& & \ddots & \\
& & & e^{j2\pi w(q+p)}
\end{vmatrix} *
$$

$$
\begin{vmatrix}
e^{-j2\pi wp} f_{q-p+1} & \cdots & e^{-j2\pi w} f_q \\
\vdots & & \vdots \\
e^{-j2\pi wp} f_q & \cdots & e^{-j2\pi w} f_{q+p}
\end{vmatrix}
$$

The diagonal determinants in the numerator and the denominator will cancel each other. Recall equation (14)

$$f_z(\tau) - \phi_1 f_z(\tau-1) - \dots - \phi_p f_z(\tau-p) = 0 \qquad \tau > q$$

Notice that the submatrix in the numerator is the same as the matrix of the denominator. The last column in the numerator (except the first entry) is a linear combination of the other columns. Multiply the first column by $e^{j2\pi wp}\phi_p$, the second by $e^{j2\pi w(p-1)}\phi_{p-1}$ and so on, then add the sum of the results to the last column . Then

$$S_p(f_{q-p+1}) =$$

$$\frac{\begin{vmatrix} 1 & \cdots & 1 & 1 - \sum_{k=1}^{p} e^{j2\pi wk}\phi_k \\ e^{-j2\pi wp} f_{q-p+1} & \cdots & e^{-j2\pi w} f_q & 0 \\ \vdots & & & 0 \\ \vdots & & & \vdots \\ e^{-j2\pi wp} f_q & \cdots & e^{-j2\pi w} f_{q+p} & 0 \end{vmatrix}}{\begin{vmatrix} e^{-j2\pi wp} f_{q-p+1} & \cdots & e^{-j2\pi w} f_q \\ \vdots & & \vdots \\ e^{-j2\pi wp} f_q & \cdots & e^{-j2\pi w} f_{q+p} \end{vmatrix}}$$

Hence

$$S_p(f_{q-p+1}) = (-1)^p \left[1 - \sum_{k=1}^{p} e^{j2\pi wk}\phi_k \right] = C_3$$

<u>Theorem 5.2 :</u>

Under the conditions of the previous theorem

$$1)\ S_n(f_m e^{j2\pi wm}) = C_4 \qquad , \qquad m \le m_1 \tag{33}$$

$$S_n(f_{m1+1} e^{j2\pi w(m1+1)}) \ne C_4$$

iff $n = p$ and $m_1 = -q-p$.

$$2)\ C_4 = \frac{-\ e^{j2\pi wp} C_1^*}{\phi_p} \tag{34}$$

<u>Proof:</u>

The constant pattern starts when $n = p$ and $m_1 = -q-p$

$$S_p(f_{-q-p}) =$$

$$\frac{\begin{vmatrix} 1 & \cdots & 1 & 1 \\ e^{j2\pi w(q-p)}f_{-q-p} & \cdots & e^{j2\pi w(-q-1)}f_{-q-1} & e^{j2\pi w(-q)}f_{-q} \\ e^{j2\pi w(-q-p+1)}f_{-q-p+1} & \cdots & e^{j2\pi w(-q)}f_{-q} & e^{j2\pi w(-q+1)}f_{-q+1} \\ \vdots & & \vdots & \vdots \\ e^{j2\pi w(-q-1)}f_{-q-1} & \cdots & e^{j2\pi w(-q+p-2)}f_{-q+p-2} & e^{j2\pi(-q+p-1)}f_{-q+p+1} \end{vmatrix}}{\begin{vmatrix} e^{j2\pi w(-q-p)}f_{-q-p} & \cdots & e^{j2\pi w(-q-1)}f_{-q-1} \\ \vdots & & \vdots \\ e^{j2\pi w(-q-1)}f_{-q-1} & \cdots & e^{j2\pi w(-q+p-2)}f_{-q+p-2} \end{vmatrix}}$$

Following the same steps in the previous proof, $S_p(f_{-q-p})$ can be written as

$$\frac{\begin{vmatrix} 1 & \cdots & 1 & 1 \\ e^{-j2\pi wp}f_{-q-p} & \cdots & e^{-j2\pi w}f_{-q-1} & f_{-q} \\ \vdots & & \vdots & \vdots \\ e^{-j2\pi wp}f_{-q+1} & \cdots & e^{-j2\pi w}f_{-q+p-2} & f_{-q+p-1} \end{vmatrix}}{\begin{vmatrix} e^{-j2\pi wp}f_{-q-p} & \cdots & e^{-j2\pi w}f_{-q-1} \\ \vdots & & \vdots \\ e^{-j2\pi wp}f_{-q+1} & \cdots & e^{-j2\pi w}f_{-q+p-2} \end{vmatrix}}$$

Notice that the submatrix which excludes the first row and last column of the numerator is the same as the denominator, also recall equation (14), the last column in the numerator (excluding the first row) is a linear combination of the other columns, hence $S_p(f_{-q-p})$ can be written as :

$$\frac{\begin{vmatrix} 1 & \cdots & 1 & C \\ e^{-j2\pi wp}f_{-q-p} & \cdots & e^{-j2\pi w}f_{-q-1} & 0 \\ \vdots & & \vdots & 0 \\ \vdots & & \vdots & \vdots \\ e^{-j2\pi wp}f_{-q+1} & \cdots & e^{-j2\pi w}f_{-q+p-2} & 0 \end{vmatrix}}{\begin{vmatrix} e^{-j2\pi wp}f_{-q-p} & \cdots & e^{-j2\pi w}f_{-q-1} \\ \vdots & & \vdots \\ e^{-j2\pi wp}f_{-q+1} & \cdots & e^{-j2\pi w}f_{-q+p-1} \end{vmatrix}}$$

Where

$$C = 1 - 1/\phi_p \left[e^{j2\pi wp} - \phi_1 e^{j2\pi w(p-1)} - \ldots - \phi_{p-1} e^{j2\pi w} \right]$$

Hence

$$S_p(f_{-q-p}) = (-1)^p\, C$$

$$S_p(-q-p) = \frac{-(-1)^p\, e^{j2\pi wp}\left[1 - \sum_{k=1}^{p} e^{-j2\pi wk}\phi_k\right]}{\phi_p}$$

$$S_p(-q-p) = \frac{-e^{j2\pi wp}\, C_1^*}{\phi_p} = C_4 \,.$$

(2) <u>Modified GPAC</u>

<u>Theorem 5.3 :</u>

Using $f_m e^{j2\pi wm}$, instead of using f_m, will modify the GPAC as follows

$$\phi^j{}_{kk}(\text{new}) = e^{j2\pi wk}\, \phi^j{}_{kk}(\text{old}) \tag{35}$$

where

$\phi^j{}_{kk}(\text{old})$ is the GPAC using f_m.

$\phi^j{}_{kk}(\text{new})$ is the GPAC using $f_m e^{j2\pi wm}$.

<u>Proof :</u>

The proof will be done by taking $\phi^j{}_{kk}$ element by element and then generalize a formula from the results .

$$\phi^0{}_{11}(\text{new}) = e^{j2\pi w} f_1/f_0 = e^{j2\pi w}(f_1/f_0) = e^{j2\pi w}\phi^0{}_{11}(\text{old})$$

$$\phi^1{}_{11}(\text{new}) = e^{j4\pi w} f_2/e^{j2\pi w} f_1 = e^{j2\pi w}(f_2/f_1) = e^{j2\pi w}\phi^1{}_{11}(\text{old})$$

$$\phi^0{}_{22}(\text{new}) = \frac{\begin{vmatrix} f_0 & e^{j2\pi w} f_1 \\ e^{j2\pi w} f_1 & e^{j4\pi w} f_2 \end{vmatrix}}{\begin{vmatrix} f_0 & e^{j2\pi w} f_1 \\ e^{j2\pi w} f_1 & f_0 \end{vmatrix}} = e^{j4\pi w}\phi^0{}_{22}(\text{old})$$

$$\phi^{1}_{22}(\text{new}) = \frac{\begin{vmatrix} e^{j2\pi w}f_1 & e^{j4\pi w}f_2 \\ e^{j4\pi w}f_1 & e^{j8\pi w}f_3 \end{vmatrix}}{\begin{vmatrix} e^{j2\pi w}f_1 & e^{j4\pi w}f_2 \\ e^{j4\pi w}f_2 & e^{j2\pi w}f_1 \end{vmatrix}} = e^{j4\pi w}\phi^{1}_{22}(\text{old})$$

Example 5.1 : (Low frequency data)

$$z_t - 1.7119z_{t-1} + .81z_{t-2} = a_t - .5a_{t-1}$$

TABLE 5.1

THE S ARRAYS OF EXAMPLE 5.1

WITH W = 0

	1	2	3	4	5
-6	-2.9874D+00	1.2111D-01	-7.5694D-02	1.0000D-06	1.0000D-06
-5	1.7346D+00	1.2111D-01	-2.2708D-01	3.4063D-01	1.0000D-06
-4	6.6200D-01	1.2111D-01	-3.0278D-02	5.1472D-01	1.0000D-06
-3	3.7064D-01	1.2111D-01	2.8299D-01	-3.1498D+14	5.0865D-17
-2	2.1273D-01	1.2111D-01	-6.0778D+14	-6.0778D+14	-4.1257D+14
-1	9.5451D-02	2.6734D-01	-7.0201D-01	1.5858D+00	-3.3605D+00
0	-8.7134D-02	1.2926D-01	-1.5844D-01	1.7602D-01	-1.8575D-01
1	-1.7542D-01	9.8100D-02	-9.8100D-02	9.8100D-02	-9.8100D-02
2	-2.7041D-01	9.8100D-02	1.2263D-01	-2.2393D-15	1.4855D+00
3	-3.9832D-01	9.8100D-02	-1.1410D-14	1.9375D+00	0.0000D+00
4	-6.3432D-01	9.8100D-02	-3.8203D-01	3.3954D+00	0.0000D+00
5	-1.5032D+00	9.8100D-02	-3.6788D-02	0.0000D+00	0.0000D+00
6	2.3217D+00	9.8100D-02	-3.4335D-01	0.0000D+00	0.0000D+00

TABLE 5.2

THE MAGNITUDE OF THE S ARRAYS OF

EXAMPLE 5.1 WITH W =.3

	1	2	3	4	5
-6	1.9291D+00	2.8128D+00	2.6046D+00	5.5427D+00	1.0000D-06
-5	3.1888D+00	2.8128D+00	4.4968D+00	7.3514D+00	1.0000D-06
-4	2.1885D+00	2.8128D+00	6.9474D+00	8.0147D+01	7.5230D-30
-3	1.9302D+00	2.8128D+00	5.1188D+00	9.7753D+15	4.9617D+15
-2	1.7945D+00	2.8128D+00	1.2171D+15	1.5683D+15	2.9356D+15
-1	1.6962D+00	3.8842D+00	8.9131D+00	1.6222D+01	3.2485D+01
0	1.5484D+00	1.8781D+00	2.0116D+00	1.8007D+00	1.7956D+00
1	1.4797D+00	2.2784D+00	2.2784D+00	2.2784D+00	2.2784D+00
2	1.4083D+00	2.2784D+00	2.8893D+00	3.2196D+00	2.5962D+00
3	1.3168D+00	2.2784D+00	3.3058D+00	3.5479D+00	3.9649D+00
4	1.1661D+00	2.2784D+00	2.2784D+00	9.7594D+00	0.0000D+00
5	9.7067D-01	2.2784D+00	3.5180D+00	6.6191D+00	0.0000D+00
6	3.7532D+00	2.2784D+00	5.7639D+00	5.9785D+00	0.0000D+00

TABLE 5.3

THE S ARRAYS OF EXAMPLE 5.1

WITH W = .5

	1	2	3	4	5
-6	9.8743D-01	4.3480D+00	1.7859D+01	3.8558D+00	1.0000D-06
-5	-3.7346D+00	4.3480D+00	3.4762D+00	1.4731D+00	1.0000D-06
-4	-2.6620D+00	4.3480D+00	1.1727D+00	2.9750D+00	-7.3211D-30
-3	-2.3706D+00	4.3480D+00	-1.2275D+00	-9.2667D+13	1.0320D+11
-2	-2.2127D+00	4.3480D+00	1.1437D+14	-4.0691D+13	7.6818D+13
-1	-2.0955D+00	5.8691D+00	9.7358D+00	2.1992D+01	4.1723D+01
0	-1.9129D+00	2.8378D+00	-2.1973D+00	2.4412D+00	-2.3063D+00
1	-1.8246D+00	3.5219D+00	-3.5219D+00	3.5219D+00	-3.5219D+00
2	-1.7296D+00	3.5219D+00	1.3724D+00	7.5628D-01	-1.5977D+00
3	-1.6017D+00	3.5219D+00	-7.5495D-01	1.1842D+00	-7.8692D-01
4	-1.3657D+00	3.5219D+00	-1.5816D+00	8.8465D-01	0.0000D+00
5	-4.9684D-01	3.5219D+00	-2.9026D+00	2.5067D+00	0.0000D+00
6	-4.3217D+00	3.5219D+00	-3.7052D+00	1.8964D+01	0.0000D+00

Notice the different values of C_3 for each value of w. In this case, when $w = .5$ C_3 is maximum. When C_3 is large the pattern of constant values will stand out clearly, and will allow easier determination of the model order. These results will be discussed in chapter eight.

CHAPTER VI

C-TABLE

This chapter will introduce the C-table which is another technique which is used to decide the order of ARMA(p,q) processes. The relationship between the C-table and the S and R arrays will also be discussed. The following are definitions and theorems concerning the C-table and how it is used for ARMA(p,q) process order determination. (Those theorems where taken from Tucker, W. T. (1982) and Lii, K. (1985)).

Definition 6.1 :

We denote L , M Pade rational approximants to the formal power series $A(x) = \sum_{j=0}^{\infty} a_j x^j$ by

$$[L/M] = \frac{P_L(x)}{Q_M(x)} \tag{36}$$

where QM(0) = 1 and $P_L(x)$ and $Q_M(x)$ have no common factors .

Theorem 6.1 :

When it exists, an [L/M] Pade approximant for A(x) is uniquely determined. Further [L/M] = $P_L(x)/Q_M(x)$ with

$$PL(x) = \begin{vmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & & \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M}x^j & \sum_{j=M-1}^{L} a_{j-M+1}x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{vmatrix} \tag{37}$$

(if the lower index on a sum exceeds the upper, the sum is set to zero) and

$$Q_M(x) = \begin{vmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & & \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M+1} & \cdots & 1 \end{vmatrix} \tag{38}$$

Definition 6.2:

By a Pade table we mean the doubly infinite array

$$\begin{matrix} [0/0] & [0/1] & [0/2] & \cdots \\ [1/0] & [1/1] & [1/2] & \cdots \\ [2/0] & [2/1] & [2/2] & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

Definition 6.3 :

$$C(L/M) = \begin{vmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_L \\ \vdots & \vdots & & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M-1} \end{vmatrix} \tag{39}$$

where $C(L/0) = 0$, $a_{-m} = a_m$, and $C(-L/M) = C(L/M)$.

Definition 6.4 :

The C-table will be constructed as follows :

$$\begin{matrix} C(0/0) & C(0/1) & C(0/2) & \cdots \\ C(1/0) & C(1/1) & C(2/2) & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

The order of the numerator q and the denominator p can be decided from the pattern of the C-table, an infinite rectangle of zeros with its upper left corner starting at q+1 and p+1 .

Example 6.1 :

$$A(x) = \frac{1 + x}{1 + x + x^2}$$

Using long division A(x) can be written as :

$$A(x) = 1 - x^2 + x^3 - x^5 + x^6 - x^8 + x^9 - x^{11} + \ldots$$

TABLE 6.1

C-TABLE OF EXAMPLE 6.1

L \ M	0	1	2	3	4	5	6
0	1	1	-1	0	-1	-1	0
1	1	0	-1	-1	1	1	-1
2	1	-1	-1	0	0	0	0
3	1	1	-1	0	0	0	0
4	1	0	-1	0	0	0	0
5	1	-1	-1	0	0	0	0

The infinite rectangle of zeros starts when $p+1 = 2$ and $q+1 = 3$, this means that the order of QM(x) = 1 and the order of PL(x) =2 .

Theorem 6.2 :

There are three possible patterns of squares of zeros : 1) there are no infinite squares of zeros, 2) there exist values of L and M (e.g. q and p), where an infinite square of zeros begins at (q+1,p+1) and due to the symmetry at (-q-1,p+1), and 3) there is a value M,(e.g. p) where an infinite square of zeros centered at (0,p+1) begins.

Theorem 6.3 :

Case (2) of threorem 6.2 occurs if and only if the { a_n } sequence is such that there exist constants ϕ_1, ϕ_2, ..., ϕ_p such that :

$$a_k = \phi_1 a_{k-1} + \phi_2 a_k-2 + \ldots + \phi_p a_{k-p} \quad k > q$$

where $p > 0$ and $\phi_p \neq 0$.

If we let $f_i = a_i$, the above equation will be exactly the same as equation (14). Hence making use of the last two theorems will give us another method to determine the order (p and q) of the ARMA(p,q) model using the C-table. The pattern will be as follows :

TABLE 6.2

C-TABLE PATTERN FOR

ARMA(p,q) PROCESS

L \ M	1	2	. . .	p	p+1	p+2	. . .	m
-l	*	*	. . .	*	0	0	. . .	0
-l+1	.	.		.	.	.		.
.	.	.		.	.	.		.
-q-1	*	*	. . .	*	0	0	. . .	0
-q	*	*	. . .	*	*	*	. . .	*
.	.	.		.	.	.		.
-1	*	*		*	*	*		*
0	*	*		*	*	*		*
1	*	*		*	*	*		*
.								
q	*	*	. . .	*	*	*	. . .	*
q+1	*	*	. . .	*	0	0	. . .	0
.	.	.		.	.	.		.
l	*	*		*	0	0		0

<u>Proof :</u>

Recall equation (14)

$$f_z(\tau) - \phi_1 f_z(\tau-1) - \dots - \phi_p f_z(\tau-p) = 0 \qquad \tau > q$$

take the case where L = q+1 and M = p+1

$$C(q+1/p+1) = \begin{vmatrix} f_{q-p+1} & \dots & f_{q+1} \\ \vdots & & \vdots \\ f_{q+1} & \dots & f_{q+p+1} \end{vmatrix}$$

the last column of the above determinant is a linear combination of the othe columns, hence

$C(q+1/p+1) = 0$

take the case where $L = q+2$ and $M = p+1$, then

$$C(q+2/p+1) = \begin{vmatrix} f_{q-p+2} & \dots & f_{q+2} \\ \vdots & & \vdots \\ f_{q+2} & \dots & f_{q+p+2} \end{vmatrix}$$

the last column of the above determinant is a linear combination of the other columns, hence

$C(q+2/p+1) = 0$

The same argument can be used to verify the whole pattern. Using the fact that $f_{-m} = f_m$, and $C(-L/M) = C(L/M)$, we will get an image of the pattern for negative L.

<u>Example 6.2 :</u>

$z_t + 1.7119z_{t-1} + .81z_{t-2} = e_t - .5e_{t-1}$

TABLE 6.3

C-TABLE FOR EXAMPLE 6.2

L \ M	0	1	2	3	4	5
-5	1.0000	1.1305	-0.1553	0.0000	0.0000	0.0000
-4	1.0000	-0.9919	-0.1918	0.0000	0.0000	0.0000
-3	1.0000	0.7007	-0.2378	0.0000	0.0000	0.0000
-2	1.0000	-0.2562	-0.2923	0.0000	0.0000	0.0000
-1	1.0000	-0.3235	-0.3609	-0.4065	0.4579	0.5158
0	1.0000	1.0000	-0.8954	-0.6714	0.3190	-0.1607
1	1.0000	-0.3235	-0.3609	-0.4065	0.4579	0.5158
2	1.0000	-0.2562	-0.2923	0.0000	0.0000	0.0000
3	1.0000	0.7007	-0.2378	0.0000	0.0000	0.0000
4	1.0000	-0.9919	-0.1918	0.0000	0.0000	0.0000
5	1.0000	1.1305	-0.1553	0.0000	0.0000	0.0000

Notice that the first column of the C-table is always ones and the second column is the autocorrelation function.

Theorem 6.4 : (Relationship between C-table and S and R arrays)

$$C(L+1/M+1) = R_{M+1}(L-M+1)S_M(L-M+1)C(L/M) \qquad (40)$$

Proof :

Using (19), 20, (21), (22), (39), and by noting that

$C(L/M) = H_M(L-M+1)$

theorem 6.4 can be easily verified.

CHAPTER VII

SINGULAR VALUE DECOMPOSITION

Another method of order determination which also uses equation (14) is the singular value decomposition (SVD) method, where the autocorrelation function will be arranged in a matrix form, the rows of this matrix coincide with equation (14). The autocorrelation function itself will be used here rather than the normalized autocorrelation function, this will create larger nonzero singular values, which will help more in determining the order of the process. In this method it will be hard to decide the order of the moving average (MA) part, so once the autoregressive order (p) is decided, the ARMA process will be considered as ARMA(p,p).

Theorem 7.1 :

Let $A \in R_p^{m \times n}$, where $R_p^{m \times n}$ is the set of all mxn matrices with rank p in R. Then there exist orthogonal (unitary) matrices $U \in R^{m \times m}$ and $V \in R^{n \times n}$ such that :

$$A = U \Sigma V^T \tag{41}$$

where

$$\Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$$

and $S = \text{diagonal}(\alpha_1, \alpha_2, \ldots, \alpha_p)$ with

$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_p > 0$

The numbers $\alpha_1, \ldots, \alpha_p$ together with $\alpha_{p+1} = 0, \ldots, \alpha_n = 0$ are called the singular values of A, and they are the positive square roots of the eigen values (which are non-negative) of AA^T.

In this chapter the Singular Value Decomposition (SVD) will be used to determine the order of ARMA(p,q). Let A be a matrix formed as follows :

$$A = \begin{bmatrix} R(0) & R(1) & R(2) & \ldots & R(n) \\ R(1) & R(0) & R(1) & \ldots & R(n-1) \\ \vdots & \vdots & \vdots & & \vdots \\ R(m) & R(m-1) & R(m-1) & \ldots & R(n-m) \end{bmatrix} \tag{42}$$

where n and m >> p and q . Also m >> n

Recall equation (14)

$R(\tau) - \phi_1 R(\tau-1) - \ldots - \phi_p R(\tau-p) = 0 \qquad \tau > q$

The procedure to find p and q will be as follows

1) Set A_0 to A, and find the SVD of A_0. (you should get n nonzero singular values).
2) set A1 to A without the first row, and find the SVD of A1.
3) Take A2 as A without the first and the second rows of A, find the SVD of A2.
4) Repeat for A3, ..., Ak.
5) Arrange the singular values for each submatrix in vector form as shown below :

TABLE 7.1

PATTERN FOR SVD OF AN ARMA(p,q) PROCESS

	0	1	...	q	q+1	q+2	...	i	i+1	...
1	α10	α11	...	α1q	α1q+1	α1q+2	...	α1i	α1i+1	...
2	α20	α21		α2q	α2q+1	α2q+2	...	α2i	α2i+1	...
	.	.		.	.	.		.	.	
	.	.		.	.	.		.	.	
	.	.		.	.	.		.	.	
p	αp0	αp1						αpi	αpi+1	...
.	.	.						0	0	...
.	.	.				αn-2q+2		.	.	
.	.	.			αn-1q+1	0		.	.	
n	αn10	αn1		αnq	0	0		.	.	

where α_{ij} is the ith singular value of A_j. p can be decided from the above pattern. Theoretically q can also be determined, but it is not practical.

Three examples will be discussed, the first is an MA(1), the second is AR(2) and the third is ARMA(2,1) .

<u>Example 7.1 :</u>

$z_t = e_t - .5e_{t-1}$

the pattern for the SVD of this process is shown in table 7.2.

TABLE 7.2

SVD PATTERN OF EXAMPLE 7.1

	0	1	2	3	4	5	6	7
1	2.9757	2.9272	2.8356	2.6535	2.2186	0.5000	0.0000	0.0000
2	2.6184	2.4527	2.1645	1.6816	0.1127	0.0000	0.0000	0.0000
3	2.1258	1.8459	1.4419	0.0280	0.0000	0.0000	0.0000	0.0000
4	1.6244	1.3222	0.0071	0.0000	0.0000	0.0000	0.0000	0.0000
5	1.2475	0.0018	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Example 7.2 :

$z_t + 1.7119z_{t-1} + .81z_{t-2} = e_t$

the SVD pattern of this process is shown in table 7.3.

TABLE 7.3

SVD PATTERN FOR EXAMPLE 7.2

	0	1	2	3	4	5
1	131.2543	123.3502	112.1698	98.2509	82.7096	67.1867
2	42.9525	38.8462	35.1439	32.3796	30.8441	30.4059
3	2.8016	1.6162	0.5961	0.0000	0.0000	0.0000
4	0.3757	0.1921	0.0000	0.0000	0.0000	0.0000
5	0.1171	0.0000	0.0000	0.0000	0.0000	0.0000

Example 7.3 :

$z_t - 1.7119z_{t-1} + .81z_{t-2} = e_t - .5e_{t-1}$

the SVD pattern is shown in table 7.4.

TABLE 7.4

SVD PATTERN FOR EXAMPLE 7.3

	0	1	2	3	4	5
1	290.4010	272.8466	248.1319	217.4467	183.2395	149.0540
2	94.0692	85.1940	77.2143	71.2102	67.7873	66.7324
3	5.9448	3.5533	1.4896	0.2962	0.0000	0.0000
4	0.6086	0.2879	0.0509	0.0000	0.0000	0.0000
5	0.0906	0.0143	0.0000	0.0000	0.0000	0.0000

CHAPTER VIII

SIMULATIONS AND COMPARISONS OF VARIOUS METHODS

The purpose of this chapter is to compare all of the methods for order determination which have been discussed in this thesis, and to illustrate their advantages and disadvantages. Six different models will be investigated, where 2000 data points will be generated for each using a random input N(0,1) to be the input e(t) of the ARMA(p,q) model, then the order p, q will be estimated using these data. Also, the effect of the number of data points will be investigated for 2000, 500, and 200 data points.

Model One

$$z_t - 1.7119z_{t-1} + .81z_{t-2} = e_t - .5e_{t-1}$$

This model is characterized by having <u>low frequency data</u>, its frequency response is shown in figure 8.1. Figure 8.2 shows the magnitude of C_3 (the constant which appears in the S array) as a function of w. It is obvious that the highest value of C3 occurs at w = .5, where at that frequency we have the lowest frequency response amplitude. This is expected because from equation (32), C_3 is the reciprocal of the frequency response of the AR part of the ARMA(p,q) process.

$$C_3 = (-1)^p \left[1 - \sum_{k=1}^{p} e^{j2\pi wk}\phi_k \right]$$

A recommended value for w for low frequency data is .5 since C_3 will have the largest value. Although the ratio between C_3 and C_4 is always constant, the pattern will be easier to recognize against a noisy background when you have large values of C_3 and C_4.

Table 8.1 - Table 8.4 show the patterns for the data of this model, Table 8.1 shows the magnitude of the S array, Table 8.2 shows the GPAC, and Table 8.3 shows the C-table, w = .5 was used. In Table 8.4 the SVD table was formed as mentioned in chapter seven.

Model Two

$z_t + 1.7119z_{t-1} + .81z_{t-2} = e_t - .5e_{t-1}$

This is a high frequency data model, the frequency response is shown in figure 8.3 and figure 8.4 shows the magnitude of C_3 versus w, where you can see that the maximum happened at w = 0 as expected (minimum frequency response amplitude correspondes to the highest C_3). See Table 8.5 - Table 8.8 for the patterns of the S array, GPAC, C-table, and SVD table, respectively.

Model Three

$z_t + .81z_{t-2} = e_t - .5e_{t-1}$

This model will generate <u>mid frequency data</u>. Figure 8.5 shows the frequency response of the model above, and since C_3 is the reciprocal of the frequency response of the AR part, C_3 will have a small value in the middle and large value for both low and high frequency (see figure 8.6), in this case w = 0 and w = .5 are both recommended. Table 8.9 -Table 8.12 show the patterns of the S array, GPAC, C-table, and SVD table, for w = .5,respectively.

Model Four

$z_t - .8z_{t-1} - .86z_{t-2} + .83z_{t-3} = e_t - .9e_{t-1}$

The data generated from this model contains a combination of low and high frequency data, the frequency response is shown in figure 8.7 and the magnitude of C_3 versus w is shown in figure 8.8, from which w = .3 is recommended. Table 8.13 -Table 8.16 show the patterns of the S array, the GPAC, the C-table, and the SVD, respectively.

Model Five

$z_t - .5z_{t-1} + .5z_{t-2} = e_t - e_{t-1}$

This model has low frequency data, so w = .5 is recommended, but it is another important point that we are illustrating with this model. In this model the constants column(ϕ_2 = .5), which should start at the p = 2 and q = 1 row, starts instead at the p = 2 column and q = 0 row. Although we are also looking for the zero pattern, still, it will be hard to decide whether a small value is a zero or not when you have a noisy data. Using the S array is decisive, C_3 and C_4 can be distinguished easily. See Table 8.17 and Table 8.18. Table 8.19 and Table 8.20 show the C-table and the SVD table.

Model Six

$z_t - .4z_{t-8} = e_t$

Figure 8.9 and figure 8.10 show the frequency response and C_3 versus w, respectively. For this model the lowest value of C_3 is .6 and the highest is 1.4, w = .3 is recommended, where $C_3 \approx 1.3$. In this model $\phi_1,\ldots,\phi_7$ are zeros, this will cause a lot of zeros in the autocorrelation function, which will cause zero strings in the GPAC. So looking for the zero row alone to decide the order of an ARMA model may be deceiving. In this case the constant pattern, the zero pattern, and the 0/0 (which will give very large numbers and very small ones) pattern, all together should help deciding the ARMA order. The patterns of this model are shown in Table 8.21 - Table 8.24.

The effect of sample size was also investigated, first using N = 2000 data points, then using 500 and 200 data points. For 2000 data points the C3 and C4 patterns were very clear and very close to the exact values. This is due to the fact the estimated autocorrelation function is very close to the exact one. For 500 and 200 data points we have a relatively poor estimate for the autocorrelation function; the pattern is still clear but the values of C3 and C4 are not as accurate as for the 2000 data points. See Table 8.25-Table 8.64.

Table 8.65 and Table 8.66 show the S array of model one using 200 noisy data points. In Table 8.65 $w = 0$, while in Table 8.66 $w = .5$. ($w = .5$ is the recommended value to be used). It is very hard to decide q using $w = 0$. On the other hand, it was decisive using $w = .5$.

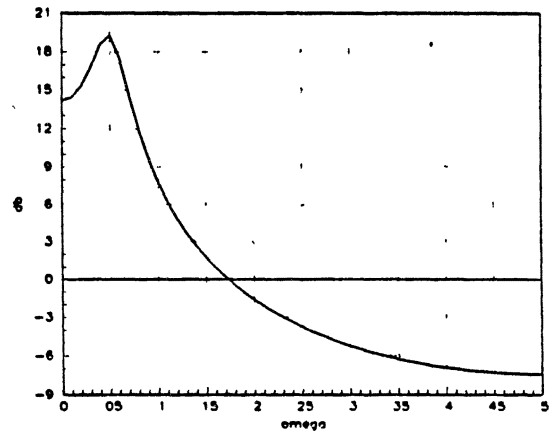

Figure 8.1 The Frequency Response of Model One

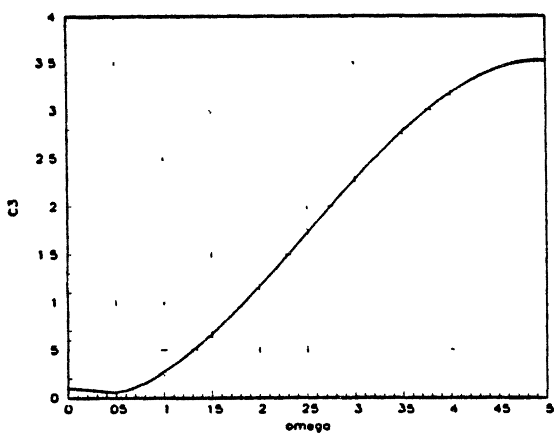

Figure 8.2 The Magnitude of C3 vs W of Model One

TABLE 8.1

THE S ARRAY OF MODEL ONE

W = .5

	1	2	3	4	5
-7	-1.2647D+00	4.3382D+00	5.6451D-01	2.4538D+00	4.1626D-01
-6	1.5624D+00	4.4022D+00	-2.8306D+00	4.0849D+00	-7.2054D+00
-5	-3.6411D+00	4.4083D+00	-2.3960D+01	8.7659D+00	-1.3021D+01
-4	-2.6737D+00	4.5364D+00	1.6129D+00	1.0775D+00	-3.8276D+01
-3	-2.3780D+00	4.2799D+00	-5.0538D-01	2.0637D+01	-3.3196D+01
-2	-2.2253D+00	4.5467D+00	-3.0486D+01	2.6360D+01	-2.9831D+01
-1	-2.1037D+00	6.0968D+00	1.2409D+01	1.9249D+01	1.3849D+02
0	-1.9060D+00	2.7729D+00	-2.2664D+00	2.5689D+00	-2.5221D+00
1	-1.8161D+00	3.3800D+00	-4.2250D+00	2.8727D+00	-8.0984D+00
2	-1.7257D+00	3.5948D+00	4.4955D-01	9.4206D-01	-4.5752D+00
3	-1.5975D+00	3.3575D+00	-9.1899D-01	1.7308D+00	-3.7560D+00
4	-1.3786D+00	3.5036D+00	-4.3219D+00	3.6502D+00	-3.9795D+00
5	-6.0974D-01	3.5267D+00	6.3272D+00	1.3004D+00	6.0045D+00
6	-4.7784D+00	3.5131D+00	-3.9930D-01	-2.6004D+00	2.4049D-01
7	-2.4890D+00	3.4679D+00	3.1987D+00	-1.6581D+00	-8.1376D+00

TABLE 8.2

THE GPAC OF MODEL ONE

W = .5

	1	2	3	4	5
0	-9.0602D-01	-4.5482D-01	1.8265D-01	-1.3346D-01	1.8211D-02
1	-8.1610D-01	-7.4340D-01	-1.3859D-01	-1.0898D-01	-2.7148D-01
2	-7.2570D-01	-8.3993D-01	8.8952D-01	-4.5648D-02	-1.3783D-01
3	-5.9749D-01	-7.4013D-01	5.6976D-01	-1.6064D+00	-9.8128D-02
4	-3.7863D-01	-7.9478D-01	-1.8038D-01	-4.1641D-01	-3.0564D-01
5	3.9026D-01	-8.0113D-01	2.2353D+00	-3.1834D-01	8.3333D-01

TABLE 8.3

THE C-TABLE OF MODEL ONE

N = 200

	0	1	2	3	4	5
-8	1.0000D+00	-2.6653D-01	-1.5026D-02	-1.4308D-04	1.4290D-06	1.8516D-08
-7	1.0000D+00	1.7900D-01	-1.9414D-02	-9.3110D-05	-5.2361D-07	-4.6381D-09
-6	1.0000D+00	-4.7374D-02	-2.3973D-02	-1.3163D-04	4.9408D-07	8.0281D-09
-5	1.0000D+00	-1.2139D-01	-2.9924D-02	-5.8888D-05	1.5520D-06	9.6338D-09
-4	1.0000D+00	3.2060D-01	-3.7650D-02	3.2647D-04	3.7271D-06	-3.1521D-08
-3	1.0000D+00	-5.3659D-01	-5.0869D-02	5.7300D-04	2.3202D-06	3.2122D-07
-2	1.0000D+00	7.3941D-01	-6.0564D-02	6.4417D-04	5.0828D-05	-2.3306D-06
-1	1.0000D+00	-9.0602D-01	-8.1469D-02	-4.6481D-03	4.6641D-04	8.5847D-06
0	1.0000D+00	1.0000D+00	-1.7912D-01	-2.5448D-02	3.4948D-03	4.7140D-04
1	1.0000D+00	-9.0602D-01	-8.1469D-02	-4.6481D-03	4.6641D-04	8.5847D-06
2	1.0000D+00	7.3941D-01	-6.0564D-02	6.4417D-04	5.0828D-05	-2.3306D-06
3	1.0000D+00	-5.3659D-01	-5.0869D-02	5.7300D-04	2.3202D-06	3.2122D-07
4	1.0000D+00	3.2060D-01	-3.7650D-02	3.2647D-04	3.7271D-06	-3.1521D-08
5	1.0000D+00	-1.2139D-01	-2.9924D-02	-5.8888D-05	1.5520D-06	9.6338D-09
6	1.0000D+00	-4.7374D-02	-2.3973D-02	-1.3163D-04	4.9408D-07	8.0281D-09
7	1.0000D+00	1.7900D-01	-1.9414D-02	-9.3110D-05	-5.2361D-07	-4.6381D-09
8	1.0000D+00	-2.6653D-01	-1.5026D-02	-1.4308D-04	1.4290D-06	1.8516D-08

TABLE 8.4

THE SVD TABLE OF MODEL ONE

N = 2000

1	3.2302D+01	3.0385D+01	2.7559D+01	2.3950D+01	1.9833D+01	1.5688D+01
2	1.1415D+01	1.0143D+01	8.9311D+00	8.0358D+00	7.5942D+00	7.5081D+00
3	1.1332D+00	6.7007D-01	3.1905D-01	3.0518D-01	1.3524D-01	1.3244D-01
4	3.8420D-01	2.7428D-01	2.5278D-01	8.3620D-02	4.6271D-02	4.6111D-02
5	2.0600D-01	8.8093D-02	4.1016D-02	2.4717D-02	2.1712D-02	2.0170D-02

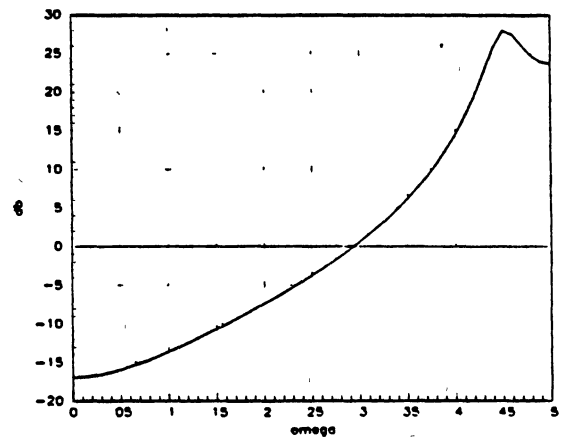

Figure 8.3 The Frequency Response of Model Two

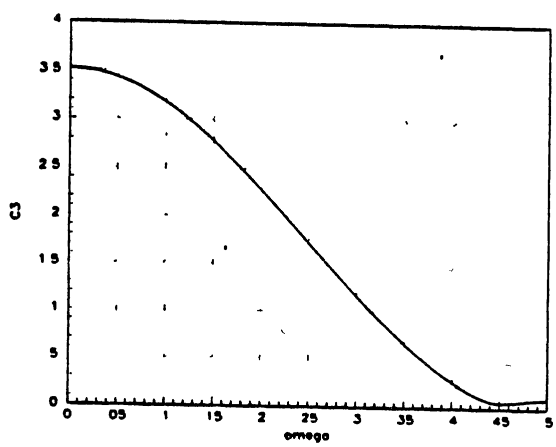

Figure 8.4 The Magnitude of C3 vs W of Model Two

TABLE 8.5

THE S ARRAY OF MODEL TWO

W = 0, N = 2000

	1	2	3	4	5
-7	-1.3053D+00	4.2553D+00	9.9562D+00	3.5883D+00	-5.9964D+00
-6	8.0871D-01	4.2501D+00	-7.6275D+00	6.7494D+00	-6.0965D+00
-5	-3.7211D+00	4.2508D+00	-9.5789D+00	1.1746D+01	-4.6795D+00
-4	-2.6308D+00	4.2565D+00	-1.0021D+01	-1.1379D+01	-7.9929D+02
-3	-2.3388D+00	4.2692D+00	-9.2916D+00	7.3600D+02	-1.5403D+03
-2	-2.1787D+00	4.2873D+00	3.5680D+01	6.0522D+01	2.7286D+01
-1	-2.0562D+00	4.1121D+00	-1.4267D+01	3.0634D+01	-8.7943D+01
0	-1.9468D+00	3.6971D+00	-4.9902D+00	5.9613D+00	-6.3948D+00
1	-1.8484D+00	3.5519D+00	-3.1849D+00	4.1672D+00	-1.9632D+00
2	-1.7470D+00	3.5676D+00	-6.6240D+00	6.3177D+00	-6.5761D+00
3	-1.6132D+00	3.5805D+00	-6.2379D+00	3.4339D+00	-4.3369D+01
4	-1.3675D+00	3.5884D+00	-6.4581D+00	4.3986D+01	-1.2818D+01
5	-4.4712D-01	3.5923D+00	-8.1144D+00	-1.9356D+01	-2.0888D+01
6	-4.2758D+00	3.5930D+00	-2.5180D+00	-2.2369D+00	-1.7812D+01
7	-2.4908D+00	3.5943D+00	8.4437D+00	7.1531D+00	-1.1500D+01

TABLE 8.6

THE GPAC OF MODEL TWO

W = 0, N = 2000

	1	2	3	4	5
0	-9.4679D-01	-8.9907D-01	-3.4976D-01	-1.9460D-01	-7.2715D-02
1	-8.4842D-01	-8.2847D-01	8.9262D-02	-6.8855D-02	7.1949D-02
2	-7.4696D-01	-8.3565D-01	-7.1290D-01	-8.5838D-03	-4.2693D-03
3	-6.1318D-01	-8.4117D-01	-6.2247D-01	3.0178D-01	-5.4259D-02
4	-3.6750D-01	-8.4417D-01	-6.7420D-01	-3.7448D+00	-2.7392D+00
5	5.5288D-01	-8.4521D-01	-1.0638D+00	2.8677D+00	-3.4262D+00

TABLE 8.7

THE C-TABLE OF MODEL TWO

N = 2000

	0	1	2	3	4	5
-8	1.0000D+00	-3.6508D-01	-2.7622D-02	1.4623D-05	1.3048D-08	-2.4597D-11
-7	1.0000D+00	2.4488D-01	-3.2675D-02	5.1692D-06	-8.3292D-09	2.0305D-11
-6	1.0000D+00	-7.4755D-02	-3.8699D-02	2.0439D-05	1.3361D-08	-6.8355D-12
-5	1.0000D+00	-1.3521D-01	-4.5786D-02	-1.9212D-05	-4.6590D-09	1.9950D-12
-4	1.0000D+00	3.6792D-01	-5.4238D-02	2.8496D-05	-1.2441D-09	-7.2834D-13
-3	1.0000D+00	-6.0002D-01	-6.4479D-02	-4.5779D-05	4.1226D-09	1.3423D-11
-2	1.0000D+00	8.0327D-01	-7.7160D-02	6.4216D-05	4.8027D-07	-3.1442D-09
-1	1.0000D+00	-9.4679D-01	-9.3135D-02	7.1941D-04	6.9752D-06	-4.3700D-08
0	1.0000D+00	1.0000D+00	-1.0359D-01	-2.0569D-03	3.5844D-05	6.0098D-07
1	1.0000D+00	-9.4679D-01	-9.3135D-02	7.1941D-04	6.9752D-06	-4.3700D-08
2	1.0000D+00	8.0327D-01	-7.7160D-02	6.4216D-05	4.8027D-07	-3.1442D-09
3	1.0000D+00	-6.0002D-01	-6.4479D-02	-4.5779D-05	4.1226D-09	1.3423D-11
4	1.0000D+00	3.6792D-01	-5.4238D-02	2.8496D-05	-1.2441D-09	-7.2834D-13
5	1.0000D+00	-1.3521D-01	-4.5786D-02	-1.9212D-05	-4.6590D-09	1.9950D-12
6	1.0000D+00	-7.4755D-02	-3.8699D-02	2.0439D-05	1.3361D-08	-6.8355D-12
7	1.0000D+00	2.4488D-01	-3.2675D-02	5.1692D-06	-8.3292D-09	2.0305D-11
8	1.0000D+00	-3.6508D-01	-2.7622D-02	1.4623D-05	1.3048D-08	-2.4597D-11

TABLE 8.8

THE SVD OF MODEL TWO

N = 2000

1	3.0619D+02	2.8921D+02	2.6476D+02	2.3424D+02	2.0059D+02	1.6819D+02
2	1.0688D+02	9.7932D+01	9.0482D+01	8.5636D+01	8.3715D+01	8.3643D+01
3	6.0780D+00	3.8888D+00	2.7021D+00	2.6339D+00	2.5517D+00	2.3724D+00
4	8.4414D-01	8.0678D-01	6.4404D-01	1.5038D-01	5.0576D-02	4.6238D-02
5	1.2645D-01	1.2182D-01	3.5219D-02	2.1830D-02	1.1004D-02	1.0830D-02

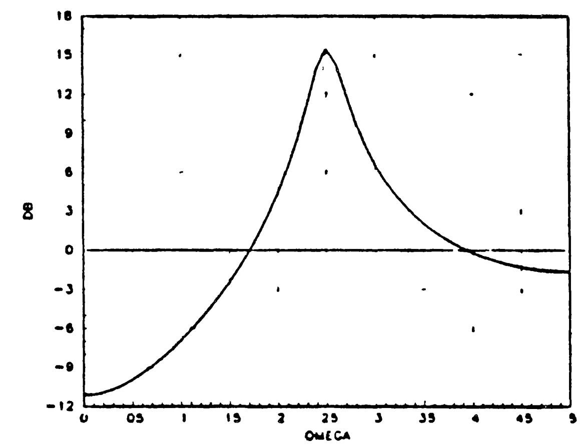

Figure 8.5 The Frequency Response of Model Three

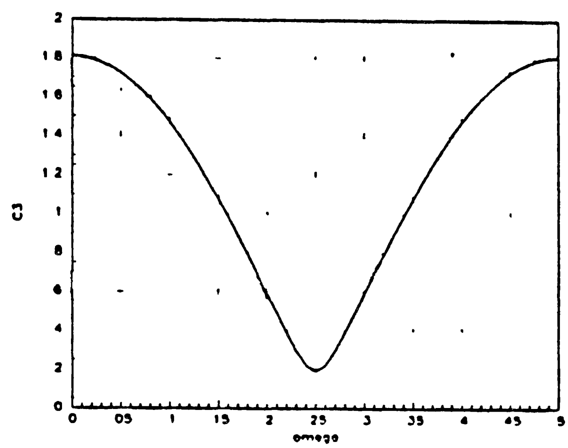

Figure 8.6 The Magnitude of C3 vs W for Model Three

TABLE 8.9

THE S ARRAY OF MODEL THREE

W = .5, N = 2000

	1	2	3	4	5
-7	4.8972D+00	2.1659D+00	-3.6385D+00	3.3922D+00	-3.3620D+00
-6	-1.1553D+00	2.1544D+00	2.5521D-01	2.4553D+00	-5.7429D+00
-5	6.7980D+00	2.1692D+00	-2.3382D+00	2.5408D+00	-5.0872D+00
-4	-1.1185D+00	2.1683D+00	2.4081D+01	3.5641D+00	-1.0600D+02
-3	9.1935D+00	2.1796D+00	-1.7212D+00	-3.2376D+01	4.4646D+01
-2	-1.0925D+00	2.1815D+00	1.7201D+02	-3.8622D+01	2.2994D+02
-1	1.2083D+01	2.0266D+00	3.3987D+00	8.9739D+00	1.3709D+01
0	-9.2356D-01	1.6968D+00	-1.1318D+00	1.2951D+00	-1.1833D+00
1	-1.1813D+01	1.8067D+00	-1.7856D+00	1.9579D+00	-1.8844D+00
2	-9.0190D-01	1.8083D+00	6.7620D+00	1.5679D+00	-1.1620D+01
3	-9.4355D+00	1.8007D+00	-1.6512D+00	3.1934D+00	-2.4463D+00
4	-8.7176D-01	1.7999D+00	-2.4769D+01	2.3640D+00	-7.8961D+00
5	-7.4406D+00	1.7906D+00	-1.8849D-01	3.9524D+00	-7.0886D+00
6	-8.3043D-01	1.7797D+00	-4.3633D+00	4.0580D+00	-1.5047D+01
7	-5.8038D+00	1.7660D+00	2.9470D-01	-1.3522D-01	6.7997D-01

TABLE 8.10

THE GPAC OF MODEL THREE

W = .5, N =2000

	1	2	3	4	5
0	7.6437D-02	-8.3726D-01	3.3300D-01	-1.4432D-01	8.6316D-02
1	-1.0813D+01	-8.2817D-01	1.0381D-02	5.0695D-02	8.1950D-03
2	9.8102D-02	-8.2964D-01	3.9286D+00	4.8427D-02	2.6026D-01
3	-8.4355D+00	-8.3046D-01	6.8570D-02	-8.9598D-01	-2.3078D-02
4	1.2824D-01	-8.2977D-01	-1.0593D+01	-9.3043D-01	-1.5522D+00
5	-6.4406D+00	-8.3114D-01	7.3859D-01	-1.6098D+00	-1.2343D+00

TABLE 8.11

THE C-TABLE OF MODEL THREE

N = 2000

	0	1	2	3	4	5
-8	1.0000D+00	4.6018D-01	-2.2192D-01	-2.4567D-03	1.9471D-06	1.1825D-07
-7	1.0000D+00	-9.5795D-02	-2.6915D-01	-2.5821D-03	4.4425D-05	7.3634D-07
-6	1.0000D+00	-5.6492D-01	-3.2754D-01	2.1532D-03	3.7136D-05	-1.6452D-07
-5	1.0000D+00	8.7712D-02	-3.9409D-01	2.9153D-03	2.3069D-05	1.3329D-07
-4	1.0000D+00	6.8398D-01	-4.7494D-01	-2.7520D-04	2.4794D-05	-8.5874D-08
-3	1.0000D+00	-8.1083D-02	-5.7190D-01	-4.0134D-03	2.7673D-05	3.7211D-06
-2	1.0000D+00	-8.2652D-01	-6.8933D-01	-1.0216D-03	-5.7143D-04	1.4297D-05
-1	1.0000D+00	7.6437D-02	-8.3236D-01	-9.8409D-02	1.1272D-02	1.7446D-03
0	1.0000D+00	1.0000D+00	-9.9416D-01	-2.9552D-01	7.8104D-02	2.0212D-02
1	1.0000D+00	7.6437D-02	-8.3236D-01	-9.8409D-02	1.1272D-02	1.7446D-03
2	1.0000D+00	-8.2652D-01	-6.8933D-01	-1.0216D-03	-5.7143D-04	1.4297D-05
3	1.0000D+00	-8.1083D-02	-5.7190D-01	-4.0134D-03	2.7673D-05	3.7211D-06
4	1.0000D+00	6.8398D-01	-4.7494D-01	-2.7520D-04	2.4794D-05	-8.5874D-08
5	1.0000D+00	8.7712D-02	-3.9409D-01	2.9153D-03	2.3069D-05	1.3329D-07
6	1.0000D+00	-5.6492D-01	-3.2754D-01	2.1532D-03	3.7136D-05	-1.6452D-07
7	1.0000D+00	-9.5795D-02	-2.6915D-01	-2.5821D-03	4.4425D-05	7.3634D-07
8	1.0000D+00	4.6018D-01	-2.2192D-01	-2.4567D-03	1.9471D-06	1.1825D-07

TABLE 8.12

THE SVD OF MODEL THREE

N = 200

1	1.3038D+01	1.1767D+01	1.1760D+01	1.0112D+01	1.0112D+01	8.2954D+00
2	1.0361D+01	1.0344D+01	9.0239D+00	9.0218D+00	7.4148D+00	7.3980D+00
3	1.5786D+00	1.1134D+00	1.0250D+00	3.1864D-01	1.5117D-01	1.5117D-01
4	1.0310D+00	7.8025D-01	1.2198D-01	1.1031D-01	9.9498D-02	9.4392D-02
5	3.5605D-01	1.0226D-01	1.0199D-01	8.7558D-02	5.6943D-02	5.5332D-02

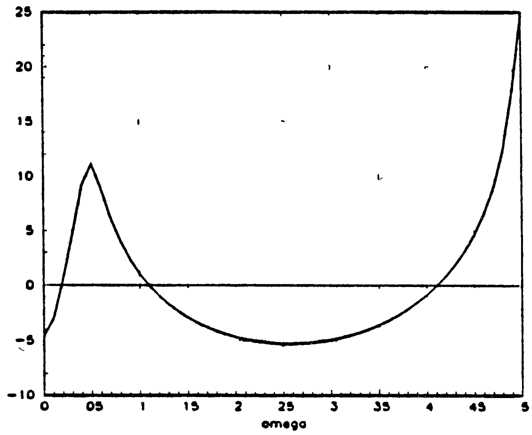

Figure 8.7 The frequency Response of Model Four

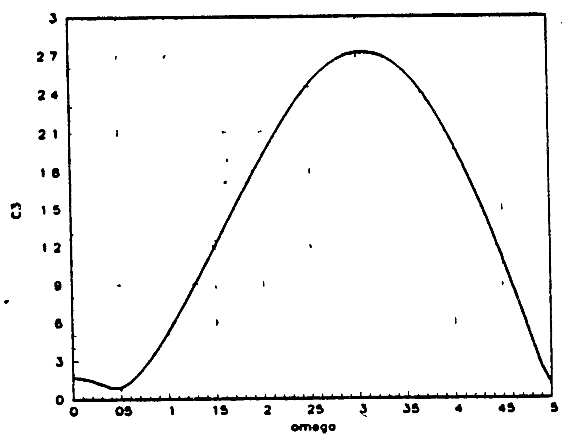

Figure 8.8 The Magnitude of C3 vs W of Model Four

TABLE 8.13

THE MAGNITUDE OF THE S ARRAY

OF MODEL FOUR

W = .3, N = 2000

	1	2	3	4	5
-7	1.0546D+00	1.6122D+00	3.3220D+00	6.4717D+01	4.4138D+00
-6	1.3350D+00	1.4009D+00	3.2583D+00	3.7940D+00	1.1214D+00
-5	1.2256D+00	2.0694D+00	3.2901D+00	3.9640D+00	3.6427D+00
-4	1.0842D+00	3.6085D+00	3.2906D+00	3.4840D+00	2.0463D+01
-3	1.5759D+00	2.6325D+00	3.2891D+00	3.1544D+01	3.8109D+02
-2	9.8749D-01	2.2802D+00	3.2611D+00	3.5062D+02	2.9161D+03
-1	2.0315D+00	2.1767D+00	7.9111D+00	1.0603D+01	1.1074D+01
0	9.6547D-01	1.6898D+00	1.8336D+00	2.0469D+00	1.7753D+00
1	1.7180D+00	1.6416D+00	2.7511D+00	2.7099D+00	2.7024D+00
2	1.0066D+00	1.5055D+00	2.7174D+00	2.8191D+00	1.6892D+01
3	1.3069D+00	1.3788D+00	2.7145D+00	1.8247D+01	7.5179D+00
4	1.1327D+00	9.9639D-01	2.7018D+00	7.5473D+00	3.6538D+01
5	1.0715D+00	4.3159D+00	2.6921D+00	3.2227D+00	7.1071D-01
6	1.3789D+00	2.3993D+00	2.6798D+00	2.7241D+00	1.8887D+01
7	9.9211D-01	2.0310D+00	2.6778D+00	2.8689D+02	1.8539D+01

TABLE 8.14

THE MAGNITUDE OF THE GPAC

OF MODEL FOUR

W = .3, N = 2000

	1	2	3	4	5
0	4.7524D-01	7.7631D-01	2.3177D-01	1.9304D-01	1.6031D-01
1	1.7398D+00	7.1993D-01	8.4361D-01	7.7290D-03	9.2670D-04
2	6.3874D-01	5.7187D-01	8.2620D-01	8.9369D-02	4.4325D-02
3	1.2054D+00	3.8209D-01	8.2492D-01	5.2373D+00	3.6739D-01
4	9.2415D-01	4.8149D-01	8.2117D-01	1.9040D+00	1.0031D+01
5	8.0260D-01	3.0807D+00	8.2624D-01	8.4943D-01	6.3379D-01

TABLE 8.15

THE MAGNITUDE OF THE C-TABLE

OF MODEL FOUR

W = .3, N = 2000

	0	1	2	3	4	5
-8	1.0000D+00	3.6513D-01	2.3414D-01	1.3978D-02	2.8375D-04	5.8314D-06
-7	1.0000D+00	6.1734D-01	2.0870D-01	1.7368D-02	3.2952D-06	1.2783D-06
-6	1.0000D+00	4.7217D-01	1.4024D-01	2.1530D-02	7.8284D-05	2.9874D-07
-5	1.0000D+00	5.8830D-01	4.5520D-02	2.6058D-02	9.2161D-05	4.7137D-07
-4	1.0000D+00	6.3658D-01	9.4539D-02	3.1733D-02	4.8404D-05	4.6993D-08
-3	1.0000D+00	5.2813D-01	2.4743D-01	3.8468D-02	9.2423D-06	1.2791D-07
-2	1.0000D+00	8.2683D-01	4.3266D-01	4.6560D-02	1.0342D-04	2.8857D-06
-1	1.0000D+00	4.7524D-01	6.0097D-01	5.5192D-02	1.3380D-02	3.1140D-03
0	1.0000D+00	1.0000D+00	7.7414D-01	2.3813D-01	6.9315D-02	1.9424D-02
1	1.0000D+00	4.7524D-01	6.0097D-01	5.5192D-02	1.3380D-02	3.1140D-03
2	1.0000D+00	8.2683D-01	4.3266D-01	4.6560D-02	1.0342D-04	2.8857D-06
3	1.0000D+00	5.2813D-01	2.4743D-01	3.8468D-02	9.2423D-06	1.2791D-07
4	1.0000D+00	6.3658D-01	9.4539D-02	3.1733D-02	4.8404D-05	4.6993D-08
5	1.0000D+00	5.8830D-01	4.5520D-02	2.6058D-02	9.2161D-05	4.7137D-07
6	1.0000D+00	4.7217D-01	1.4024D-01	2.1530D-02	7.8284D-05	2.9874D-07
7	1.0000D+00	6.1734D-01	2.0870D-01	1.7368D-02	3.2952D-06	1.2783D-06
8	1.0000D+00	3.6513D-01	2.3414D-01	1.3978D-02	2.8375D-04	5.8314D-06

TABLE 8.16

THE SVD OF MODEL FOUR

N =2000

1	2.8539D+01	2.7556D+01	2.6587D+01	2.5435D+01	2.4244D+01	2.2992D+01
2	5.3761D+00	5.3098D+00	5.2097D+00	4.9622D+00	4.7037D+00	4.2403D+00
3	3.9776D+00	3.7153D+00	3.3216D+00	3.0875D+00	2.7780D+00	2.6909D+00
4	2.3789D+00	1.9263D+00	1.6687D+00	1.1902D+00	8.8181D-01	6.7417D-01
5	7.4507D-01	7.2425D-01	4.9074D-01	4.8706D-01	4.6427D-01	2.3786D-01
6	7.1034D-01	4.9074D-01	4.8704D-01	4.3012D-01	2.1886D-01	4.4646D-02
7	3.9496D-01	3.8920D-01	3.0613D-01	1.4092D-01	2.8651D-02	2.6038D-02
8	3.7592D-01	2.2288D-01	1.1945D-01	2.6743D-02	2.5361D-02	2.3775D-02

TABLE 8.17

THE S ARRAY OF MODEL FIVE

W = .5, N = 2000

	1	2	3	4	5
-7	-7.2197D-01	3.4035D+00	-1.7108D+00	1.7624D+00	-3.1040D-03
-6	-8.2478D+00	3.6524D+00	-2.1736D+00	8.8388D+00	2.3603D+00
-5	-1.6385D+00	3.8465D+00	2.2801D+00	-5.8260D+00	-3.5104D+00
-4	1.1324D+00	3.6049D+00	6.8416D+00	-1.5786D+01	1.8816D+01
-3	-2.6245D+00	3.4481D+00	6.6626D-01	-1.9748D+01	-1.9776D+02
-2	-9.4267D-01	4.0185D+00	1.5100D+01	-1.7842D+01	6.1673D+01
-1	-3.5325D+01	3.0491D+00	2.6161D+00	5.4535D+00	4.6265D+00
0	-1.0291D+00	1.5534D+00	-9.7468D-01	1.1868D+00	-9.4450D-01
1	1.6444D+01	2.1107D+00	-1.7561D+00	2.0578D+00	-1.8906D+00
2	-1.6156D+00	2.2234D+00	-3.1621D-01	1.3437D+00	-1.2325D+00
3	-5.3105D-01	1.9527D+00	-1.2984D+00	1.3556D+00	2.7369D+01
4	-2.5661D+00	1.9825D+00	-7.6808D-01	-4.4061D+00	8.0154D-01
5	-1.1380D+00	2.0638D+00	2.6817D+00	-2.1318D+00	-1.4518D+00
6	2.5967D+00	1.8244D+00	1.6075D+00	6.8887D+00	2.4200D-03
7	-1.4374D+00	1.5906D+00	-2.0159D-01	1.3716D+00	1.2004D+00

TABLE 8.18

THE GPAC OF MODEL FIVE

W = .5, N = 2000

	1	2	3	4	5
0	-2.9133D-02	-5.0947D-01	3.7257D-01	-2.1762D-01	2.0415D-01
1	1.7444D+01	-5.2524D-01	1.1630D-01	1.1534D-01	3.0656D-02
2	-6.1557D-01	-6.4483D-01	4.7461D-01	6.8043D-02	-6.2324D-03
3	4.6895D-01	-5.4168D-01	1.8977D-01	8.5869D-02	-1.4546D+00
4	-1.5661D+00	-5.1540D-01	3.3687D-01	-7.5628D-01	2.2833D-01
5	-1.3797D-01	-5.6506D-01	1.2337D+00	2.4119D-01	6.1511D-01

TABLE 8.19

THE C-TABLE OF MODEL FIVE

N = 2000

	0	1	2	3	4	5
-8	1.0000D+00	-4.9869D-02	-3.0927D-03	-4.5973D-04	3.3676D-05	2.5916D-06
-7	1.0000D+00	1.1401D-01	-1.4579D-02	-1.1265D-03	4.9241D-05	1.7758D-06
-6	1.0000D+00	3.1699D-02	-2.7198D-02	-1.1989D-03	1.2598D-05	2.2777D-06
-5	1.0000D+00	-2.2975D-01	-4.8133D-02	-9.7175D-04	-5.2232D-05	3.7029D-06
-4	1.0000D+00	1.4670D-01	-9.3390D-02	-2.8847D-03	-6.9065D-05	1.6217D-05
-3	1.0000D+00	3.1282D-01	-1.7241D-01	-1.5201D-02	8.0431D-04	-1.1149D-05
-2	1.0000D+00	-5.0819D-01	-2.6737D-01	-3.2028D-02	-1.1821D-02	1.7889D-03
-1	1.0000D+00	-2.9133D-02	-5.0904D-01	-2.7540D-01	1.0249D-01	5.8354D-02
0	1.0000D+00	1.0000D+00	-9.9915D-01	-7.3918D-01	4.7095D-01	2.8584D-01
1	1.0000D+00	-2.9133D-02	-5.0904D-01	-2.7540D-01	1.0249D-01	5.8354D-02
2	1.0000D+00	-5.0819D-01	-2.6737D-01	-3.2028D-02	-1.1821D-02	1.7889D-03
3	1.0000D+00	3.1282D-01	-1.7241D-01	-1.5201D-02	8.0431D-04	-1.1149D-05
4	1.0000D+00	1.4670D-01	-9.3390D-02	-2.8847D-03	-6.9065D-05	1.6217D-05
5	1.0000D+00	-2.2975D-01	-4.8133D-02	-9.7175D-04	-5.2232D-05	3.7029D-06
6	1.0000D+00	3.1699D-02	-2.7198D-02	-1.1989D-03	1.2598D-05	2.2777D-06
7	1.0000D+00	1.1401D-01	-1.4579D-02	-1.1265D-03	4.9241D-05	1.7758D-06
8	1.0000D+00	-4.9869D-02	-3.0927D-03	-4.5973D-04	3.3676D-05	2.5916D-06

TABLE 8.20

THE SVD OF MODEL FIVE

N=2000

1	4.2003D+00	4.0096D+00	3.7558D+00	3.1682D+00	3.0071D+00	1.8093D+00
2	3.9166D+00	3.5234D+00	3.0706D+00	2.8185D+00	1.7426D+00	1.5901D+00
3	1.6291D+00	1.2832D+00	1.2644D+00	5.5581D-01	1.7990D-01	1.7765D-01
4	1.2368D+00	1.1178D+00	3.1079D-01	1.4243D-01	1.3969D-01	1.2413D-01
5	7.4368D-01	2.2141D-01	1.3280D-01	1.3268D-01	9.2250D-02	8.9789D-02

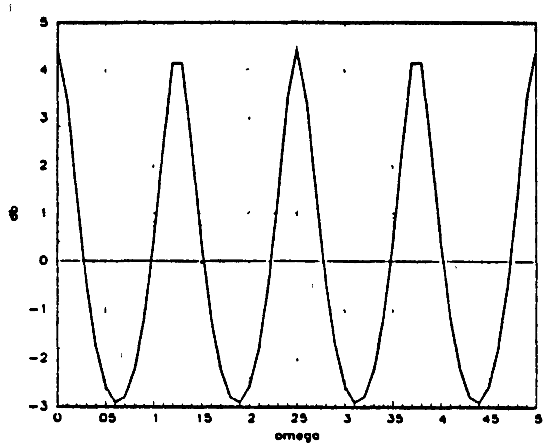

Figure 8.9 The frequency Response of Model Six

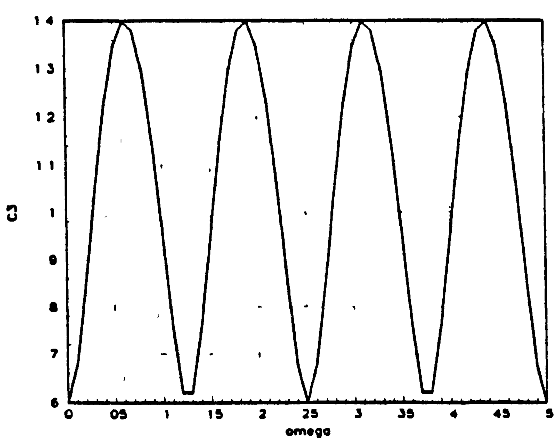

Figure 8.10 The Magnitude of C3 vs W of Model Six

TABLE 8.21

THE MAGNITUDE OF THE S ARRAY

OF MODEL SIX

W = .3, N = 2000

	1	2	...	7	8	9	10
-7	2.7735D+01	9.7467D-01		4.1942D+01	3.2048D+00	7.5294D+00	1.1289D+01
-6	9.6093D-01	6.9813D+00		3.5724D+01	3.3393D+00	3.7419D+00	5.2555D+00
-5	3.5539D+01	6.9763D+00		1.0312D+03	3.2494D+00	8.4453D+00	5.6757D+00
-4	9.9764D-01	1.6757D+00		2.6728D+01	3.2517D+00	3.2907D+00	3.1792D+00
-3	9.8123D+01	1.8021D+00		1.7625D+03	3.2573D+00	6.0478D+00	2.4231D+01
-2	1.0555D+00	3.4735D+01	...	2.4452D+02	3.2657D+00	2.4527D+01	9.2167D+01
-1	4.4198D+01	3.5299D+01		2.3758D+04	3.3052D+00	7.9634D+01	8.1206D+02
0	9.9329D-01	1.0173D+00		1.0019D+00	1.3531D+00	1.3542D+00	1.3563D+00
1	1.3765D+00	1.0278D+00		9.7260D+03	1.3370D+00	1.4030D+00	1.4819D+00
2	9.9691D-01	2.3916D+00		1.0105D+02	1.3336D+00	1.2992D+00	3.4870D+01
3	1.2931D+02	2.2063D+00		7.2043D+02	1.3313D+00	4.2212D+01	2.0722D+01
4	9.9173D-01	1.1292D+00		1.0806D+01	1.3206D+00	1.6656D+00	2.8352D+00
5	5.5991D+00	1.1336D+00		4.2034D+02	1.3570D+00	4.5305D+00	4.2073D+00
6	9.8956D-01	6.4069D+01		1.3754D+01	1.2921D+00	1.2804D+00	9.8624D-01
7	1.8308D+03	6.6155D+01		1.7834D+01	1.3245D+00	1.3998D+00	9.1321D-01

TABLE 8.22

THE MAGNITUDE OF THE GPAC

OF MODEL SIX

W = .3, N = 2000

	1	2	...	7	8	9	10
0	2.2473D-02	2.8819D-02		4.2170D-05	4.0940D-01	1.7005D-02	1.6702D-03
1	1.3042D+00	2.9591D-02		3.9777D+01	4.0940D-01	5.7203D-02	1.6079D-02
2	1.0160D-02	1.3271D+00	...	5.7333D-02	4.0942D-01	2.1482D-01	1.4391D+00
3	1.2962D+02	1.3167D+00		2.6955D+01	4.0942D-01	1.2828D+01	6.5179D+00
4	2.7906D-02	1.6186D-01		1.0479D-02	4.0642D-01	1.9723D-01	4.9953D-01
5	5.8268D+00	1.6238D-01		1.1766D+01	4.0638D-01	1.2108D+00	8.0056D-01
6	3.5678D-02	6.5733D+01		3.2794D-01	4.0316D-01	1.7005D-01	8.7362D-02
7	1.8311D+03	6.5331D+01		1.7868D+01	4.0207D-01	6.6168D-02	1.6941D-01
8	7.8178D-02	4.7574D-02		3.7081D-02	3.9409D-01	1.3638D+00	1.8664D-01

TABLE 8.23

THE MAGNITUDE OF THE C- TABLE

OF MODEL SIX

W = .3, N = 2000

	0	1	•••	6	7	8	9
-7	1.0000D+00	2.2391D-04		3.2305D-04	1.0362D-04	1.8442D-03	8.9070D-05
-6	1.0000D+00	6.2758D-03		2.2435D-05	3.1597D-04	4.5742D-03	5.2379D-04
-5	1.0000D+00	1.0771D-03		7.1871D-05	2.6854D-05	1.1256D-02	4.3261D-04
-4	1.0000D+00	3.8597D-02		2.3701D-04	2.5626D-03	2.7696D-02	2.1935D-03
-3	1.0000D+00	2.9778D-04	•••	6.2681D-05	9.5070D-05	6.7647D-02	1.7100D-04
-2	1.0000D+00	2.9309D-02		1.6665D-05	1.6582D-03	1.6523D-01	7.9601D-04
-1	1.0000D+00	2.2473D-02		4.0616D-03	4.1688D-05	4.0359D-01	1.3916D-02
0	1.0000D+00	1.0000D+00		9.9132D-01	9.8857D-01	9.8582D-01	8.1830D-01
1	1.0000D+00	2.2473D-02		4.0616D-03	4.1688D-05	4.0359D-01	1.3916D-02
2	1.0000D+00	2.9309D-02		1.6665D-05	1.6582D-03	1.6523D-01	7.9601D-04
3	1.0000D+00	2.9778D-04		6.2681D-05	9.5070D-05	6.7647D-02	1.7100D-04
4	1.0000D+00	3.8597D-02		2.3701D-04	2.5626D-03	2.7696D-02	2.1935D-03
5	1.0000D+00	1.0771D-03		7.1871D-05	2.6854D-05	1.1256D-02	4.3261D-04
6	1.0000D+00	6.2758D-03		2.2435D-05	3.1597D-04	4.5742D-03	5.2379D-04
7	1.0000D+00	2.2391D-04		3.2305D-04	1.0362D-04	1.8442D-03	8.9070D-05

TABLE 8.24

THE SVD OF MODEL SIX

N = 2000

1	1.8737D+00	1.8459D+00	1.7951D+00	1.7433D+00	1.5336D+00	1.5198D+00
2	1.7253D+00	1.6831D+00	1.6771D+00	1.4858D+00	1.4064D+00	1.3944D+00
3	1.6640D+00	1.6505D+00	1.4274D+00	1.3673D+00	1.3352D+00	1.3344D+00
4	1.6242D+00	1.3987D+00	1.3551D+00	1.3236D+00	1.3118D+00	1.3048D+00
5	1.3060D+00	1.2807D+00	1.2712D+00	1.2700D+00	1.2676D+00	1.2624D+00
6	1.2430D+00	1.2410D+00	1.2391D+00	1.2391D+00	1.2339D+00	1.1959D+00
7	1.2029D+00	1.2001D+00	1.1913D+00	1.1797D+00	1.1736D+00	1.1731D+00
8	1.1882D+00	1.1787D+00	1.1741D+00	1.1729D+00	1.1704D+00	4.7602D-01
9	7.3511D-01	7.2712D-01	7.1529D-01	7.0806D-01	7.8956D-02	7.3504D-02
10	7.0928D-01	7.0330D-01	7.0153D-01	7.7666D-02	6.9062D-02	6.9050D-02
11	6.9626D-01	6.9522D-01	7.0904D-02	6.2867D-02	5.6407D-02	5.4245D-02
12	6.9409D-01	7.0854D-02	6.2867D-02	5.4849D-02	5.3056D-02	5.1665D-02

TABLE 8.25

THE S ARRAY OF MODEL ONE

W = .5, N = 500

	1	2	3	4	5
-7	-1.0468D+00	4.5315D+00	-1.3608D+01	1.8291D+01	-8.0921D+00
-6	2.5526D+01	4.5179D+00	1.0642D+01	9.0236D+00	4.5399D+01
-5	-3.2255D+00	4.4793D+00	1.9796D-01	-5.8979D+00	3.6192D+00
-4	-2.5910D+00	4.5822D+00	5.5551D+00	-6.0288D+00	-9.2399D+01
-3	-2.3448D+00	4.2792D+00	-2.3875D-01	1.9585D+01	-9.2362D+00
-2	-2.2081D+00	4.6030D+00	-2.1609D+01	2.0128D+01	-1.9218D+01
-1	-2.0925D+00	5.7892D+00	1.4276D+01	2.0495D+01	-2.3941D+03
0	-1.9154D+00	2.8624D+00	-2.3843D+00	2.6982D+00	-2.7012D+00
1	-1.8278D+00	3.3512D+00	-4.5776D+00	2.6781D+00	4.0304D+01
2	-1.7436D+00	3.6025D+00	1.9708D-01	1.9072D+00	1.6174D+00
3	-1.6285D+00	3.3352D+00	-1.8840D+00	1.8122D+00	-2.2996D+00
4	-1.4493D+00	3.4380D+00	-1.4238D-01	2.4497D+00	3.6795D+00
5	-9.6230D-01	3.3640D+00	-2.3675D+00	2.3166D+00	-2.6625D+00
6	-2.2371D+01	3.3726D+00	-5.0487D+00	3.1918D+00	2.7730D+01
7	-2.5935D+00	3.3675D+00	-8.1710D-01	3.4949D+01	-6.1578D+00

TABLE 8.26

THE GPAC OF MODEL ONE

W = .5 , N = 500

	1	2	3	4	5
0	-9.1535D-01	-4.9443D-01	1.6702D-01	-1.3165D-01	-1.1283D-03
1	-8.2778D-01	-7.2805D-01	-2.1184D-01	-1.3305D-01	2.0972D+00
2	-7.4363D-01	-8.4185D-01	8.2545D-01	-9.7381D-02	1.7511D-01
3	-6.2855D-01	-7.2787D-01	3.3914D-01	3.0059D-01	-2.4888D-02
4	-4.4933D-01	-7.6752D-01	7.1922D-01	4.1535D-01	-1.0166D+00
5	3.7698D-02	-7.4459D-01	2.2247D-01	-2.5673D-01	5.8648D-02

TABLE 8.27

THE C-TABLE OF MODEL ONE

N = 500

	0	1	2	3	4	5
-8	1.0000D+00	-2.0429D-01	-1.1247D-02	-6.5721D-06	5.6592D-08	-9.1778D-10
-7	1.0000D+00	1.2820D-01	-1.5211D-02	-1.1677D-05	2.2564D-08	-5.8307D-10
-6	1.0000D+00	-5.9991D-03	-2.0438D-02	3.1474D-05	1.2930D-07	-1.7015D-10
-5	1.0000D+00	-1.5913D-01	-2.7448D-02	1.4148D-04	5.0366D-07	-2.9013D-09
-4	1.0000D+00	3.5416D-01	-3.5762D-02	1.9671D-04	-1.2126D-06	2.8538D-09
-3	1.0000D+00	-5.6345D-01	-4.9132D-02	5.8002D-04	4.0341D-06	-1.1466D-07
-2	1.0000D+00	7.5771D-01	-5.8362D-02	7.0267D-04	4.1426D-05	-6.5481D-07
-1	1.0000D+00	-9.1535D-01	-8.0163D-02	-3.3171D-03	3.1135D-04	-3.1224D-07
0	1.0000D+00	1.0000D+00	-1.6213D-01	-1.9861D-02	2.3650D-03	2.7674D-04
1	1.0000D+00	-9.1535D-01	-8.0163D-02	-3.3171D-03	3.1135D-04	-3.1224D-07
2	1.0000D+00	7.5771D-01	-5.8362D-02	7.0267D-04	4.1426D-05	-6.5481D-07
3	1.0000D+00	-5.6345D-01	-4.9132D-02	5.8002D-04	4.0341D-06	-1.1466D-07
4	1.0000D+00	3.5416D-01	-3.5762D-02	1.9671D-04	-1.2126D-06	2.8538D-09
5	1.0000D+00	-1.5913D-01	-2.7448D-02	1.4148D-04	5.0366D-07	-2.9013D-09
6	1.0000D+00	-5.9991D-03	-2.0438D-02	3.1474D-05	1.2930D-07	-1.7015D-10
7	1.0000D+00	1.2820D-01	-1.5211D-02	-1.1677D-05	2.2564D-08	-5.8307D-10
8	1.0000D+00	-2.0429D-01	-1.1247D-02	-6.5721D-06	5.6592D-08	-9.1778D-10

TABLE 8.28

THE SVD TABLE OF MODEL ONE

N = 500

1	4.1544D+01	3.8953D+01	3.5196D+01	3.0428D+01	2.4975D+01	1.9385D+01
2	1.3563D+01	1.1861D+01	1.0163D+01	8.7685D+00	7.9010D+00	7.5496D+00
3	1.4567D+00	8.7394D-01	3.6794D-01	2.4418D-01	1.6763D-01	1.5929D-01
4	4.1818D-01	2.7827D-01	1.6627D-01	1.1443D-01	7.2876D-02	7.1600D-02
5	2.1746D-01	7.3757D-02	6.1033D-02	3.9261D-02	3.3916D-02	3.1834D-02

TABLE 8.29

THE S ARRAY OF MODEL TWO

W = 0, N = 500

	1	2	3	4	5
-7	-1.1195D+00	4.4918D+00	-8.2339D+00	2.6979D+04	8.8687D+00
-6	7.8628D+00	4.5088D+00	-3.2800D-01	1.0414D+01	-6.7058D+00
-5	-3.3368D+00	4.5189D+00	-1.0772D+01	1.0535D+01	-4.8790D+00
-4	-2.6164D+00	4.5583D+00	-5.3086D+00	1.3505D+01	-1.7630D+02
-3	-2.3569D+00	4.5671D+00	1.9948D+01	4.3233D+01	3.1936D+01
-2	-2.1954D+00	4.5165D+00	-3.6052D+01	9.5475D+01	-9.3019D+00
-1	-2.0624D+00	4.1862D+00	-1.2160D+01	3.2298D+01	-9.7779D+01
0	-1.9413D+00	3.6199D+00	-5.1542D+00	6.1330D+00	-6.5434D+00
1	-1.8365D+00	3.3799D+00	-3.8239D+00	4.3820D+00	1.7725D+00
2	-1.7370D+00	3.3448D+00	-2.7273D+00	1.6421D+01	-4.1159D+00
3	-1.6187D+00	3.3517D+00	-2.3997D+01	9.6877D+00	-1.2835D+01
4	-1.4279D+00	3.3916D+00	-5.8796D+00	1.1852D+01	6.7037D+00
5	-8.8717D-01	3.4131D+00	2.6713D-01	7.7710D+00	1.3610D+01
6	-9.3652D+00	3.4049D+00	-7.5262D+00	7.5263D+00	-3.4616D+00
7	-2.5563D+00	3.4205D+00	-7.5240D+00	1.0534D+04	-7.8134D+00

TABLE 8.30

THE GPAC OF MODEL TWO

W = 0, N = 500

	1	2	3	4	5
0	-9.4126D-01	-8.6472D-01	-4.2386D-01	-1.8989D-01	-6.6920D-02
1	-8.3651D-01	-7.4833D-01	-1.0607D-01	-4.5897D-02	1.9055D-01
2	-7.3695D-01	-7.3237D-01	1.3672D-01	-3.7982D-01	1.2888D-01
3	-6.1868D-01	-7.3530D-01	-4.5205D+00	-7.1735D-01	-7.2802D-02
4	-4.2793D-01	-7.5054D-01	-5.4581D-01	-1.1250D+00	1.3740D+00
5	1.1283D-01	-7.5699D-01	8.1442D-01	-7.4619D-01	2.0296D+00

TABLE 8.31

THE C-TABLE OF MODEL TWO

N = 500

	0	1	2	3	4	5
-8	1.0000D+00	-2.2566D-01	-1.3210D-02	-3.3828D-05	5.5022D-08	8.9442D-11
-7	1.0000D+00	1.4500D-01	-1.7113D-02	3.7029D-05	4.3040D-11	2.2925D-10
-6	1.0000D+00	-1.7333D-02	-2.2575D-02	-4.0510D-05	1.5428D-07	5.8735D-10
-5	1.0000D+00	-1.5362D-01	-2.9822D-02	-4.9741D-05	2.0676D-07	2.8940D-10
-4	1.0000D+00	3.5899D-01	-3.9734D-02	9.1132D-05	1.8378D-07	2.1063D-10
-3	1.0000D+00	-5.8026D-01	-5.4038D-02	-2.0160D-05	2.5619D-07	-2.8931D-09
-2	1.0000D+00	7.8738D-01	-7.3785D-02	-1.4745D-04	6.7450D-07	-2.2448D-08
-1	1.0000D+00	-9.4126D-01	-9.8600D-02	1.3902D-03	1.4696D-05	-1.1781D-07
0	1.0000D+00	1.0000D+00	-1.1403D-01	-3.2799D-03	7.7394D-05	1.7604D-06
1	1.0000D+00	-9.4126D-01	-9.8600D-02	1.3902D-03	1.4696D-05	-1.1781D-07
2	1.0000D+00	7.8738D-01	-7.3785D-02	-1.4745D-04	6.7450D-07	-2.2448D-08
3	1.0000D+00	-5.8026D-01	-5.4038D-02	-2.0160D-05	2.5619D-07	-2.8931D-09
4	1.0000D+00	3.5899D-01	-3.9734D-02	9.1132D-05	1.8378D-07	2.1063D-10
5	1.0000D+00	-1.5362D-01	-2.9822D-02	-4.9741D-05	2.0676D-07	2.8940D-10
6	1.0000D+00	-1.7333D-02	-2.2575D-02	-4.0510D-05	1.5428D-07	5.8735D-10
7	1.0000D+00	1.4500D-01	-1.7113D-02	3.7029D-05	4.3040D-11	2.2925D-10
8	1.0000D+00	-2.2566D-01	-1.3210D-02	-3.3828D-05	5.5022D-08	8.9442D-11

TABLE 8.32

THE SVD OF MODEL TWO

N = 500

1	2.0433D+02	1.9159D+02	1.7306D+02	1.4952D+02	1.2281D+02	9.5682D+01
2	6.8633D+01	6.0776D+01	5.3162D+01	4.6957D+01	4.2914D+01	4.1023D+01
3	6.4172D+00	4.2149D+00	2.5012D+00	1.9518D+00	1.9027D+00	1.8733D+00
4	7.4871D-01	5.9958D-01	5.9226D-01	3.7580D-01	3.5784D-01	3.4029D-01
5	1.4078D-01	1.3965D-01	9.2482D-02	8.4063D-02	3.5620D-02	3.0594D-02

TABLE 8.33

THE S ARRAY OF MODEL THREE

W = .5, N = 500

	1	2	3	4	5
-7	1.7029D+01	2.1799D+00	4.6259D-01	3.8560D-01	-7.4079D+01
-6	-1.0935D+00	2.2079D+00	-6.0733D-01	2.9790D+00	-5.1083D+00
-5	1.1590D+01	2.1877D+00	-8.1741D+00	6.6462D+00	-7.4495D+00
-4	-1.1032D+00	2.2337D+00	-1.2174D+00	1.3558D+00	-1.5992D+01
-3	1.0893D+01	2.2128D+00	-1.4763D+00	-1.0109D+02	-1.0622D+01
-2	-1.0899D+00	2.2071D+00	-6.7030D+01	2.0409D+01	-1.3396D+01
-1	1.2586D+01	2.0440D+00	3.4541D+00	8.2894D+00	3.1628D+01
0	-9.2639D-01	1.6943D+00	-1.1367D+00	1.3174D+00	-1.2647D+00
1	-1.2127D+01	1.8094D+00	-1.8663D+00	1.5962D+00	-4.1874D+00
2	-9.1592D-01	1.8048D+00	3.6461D+00	-2.1924D+00	-2.3794D+00
3	-1.0690D+01	1.8189D+00	2.2244D+00	2.5024D+01	-3.4181D+00
4	-9.2057D-01	1.8570D+00	-2.5553D+00	3.1283D+00	-3.8243D+00
5	-1.1695D+01	1.8714D+00	7.1915D-01	4.1237D-01	1.3697D+00
6	-9.4454D-01	1.8952D+00	-3.2830D-01	-5.7094D-01	5.9486D-01
7	-1.6701D+01	1.8754D+00	1.2559D+00	-5.9176D-01	4.3160D+01

TABLE 8.34

THE GPAC OF MODEL THREE

W = .5, N = 500

	1	2	3	4	5
0	7.3607D-02	-8.2893D-01	3.2910D-01	-1.5892D-01	3.9986D-02
1	-1.1127D+01	-8.1979D-01	-2.7843D-02	-7.8208D-02	-3.1260D-01
2	8.4081D-02	-8.1562D-01	2.4697D+00	-2.1688D-02	-2.2401D-01
3	-9.6896D+00	-8.1432D-01	1.8272D+00	-1.8456D+01	-2.1373D-01
4	7.9425D-02	-8.4883D-01	-3.1261D-01	-4.7069D-01	-5.1337D-01
5	-1.0695D+01	-8.4759D-01	1.1841D+00	-1.3843D-01	2.6813D-01

TABLE 8.35

THE C-TABLE OF MODEL THREE

N = 500

	0	1	2	3	4	5
-8	1.0000D+00	4.9362D-01	-2.4445D-01	-4.8292D-03	5.9175D-05	1.6962D-06
-7	1.0000D+00	-3.1439D-02	-2.8079D-01	-3.3620D-03	-4.1219D-05	1.5365D-08
-6	1.0000D+00	-5.6683D-01	-3.2296D-01	-4.7371D-03	2.7838D-05	1.9134D-06
-5	1.0000D+00	5.2998D-02	-3.8104D-01	-4.0005D-03	2.0110D-04	7.1360D-06
-4	1.0000D+00	6.6727D-01	-4.4889D-01	1.2797D-02	4.2725D-04	-1.3900D-05
-3	1.0000D+00	-6.8864D-02	-5.5125D-01	7.0038D-03	2.3149D-05	6.5036D-05
-2	1.0000D+00	-8.1902D-01	-6.7586D-01	2.8359D-03	1.0674D-03	-2.9033D-04
-1	1.0000D+00	7.3607D-02	-8.2444D-01	-1.0185D-01	1.3648D-02	9.2878D-04
0	1.0000D+00	1.0000D+00	-9.9458D-01	-3.0949D-01	8.5878D-02	2.3227D-02
1	1.0000D+00	7.3607D-02	-8.2444D-01	-1.0185D-01	1.3648D-02	9.2878D-04
2	1.0000D+00	-8.1902D-01	-6.7586D-01	2.8359D-03	1.0674D-03	-2.9033D-04
3	1.0000D+00	-6.8864D-02	-5.5125D-01	7.0038D-03	2.3149D-05	6.5036D-05
4	1.0000D+00	6.6727D-01	-4.4889D-01	1.2797D-02	4.2725D-04	-1.3900D-05
5	1.0000D+00	5.2998D-02	-3.8104D-01	-4.0005D-03	2.0110D-04	7.1360D-06
6	1.0000D+00	-5.6683D-01	-3.2296D-01	-4.7371D-03	2.7838D-05	1.9134D-06
7	1.0000D+00	-3.1439D-02	-2.8079D-01	-3.3620D-03	-4.1219D-05	1.5365D-08
8	1.0000D+00	4.9362D-01	-2.4445D-01	-4.8292D-03	5.9175D-05	1.6962D-06

TABLE 8.36

THE SVD OF MODEL THREE

N = 500

1	1.3074D+01	1.1816D+01	1.1810D+01	1.0169D+01	1.0168D+01	8.3642D+00
2	1.0385D+01	1.0371D+01	9.0516D+00	9.0493D+00	7.4487D+00	7.4379D+00
3	1.7396D+00	1.2204D+00	1.1835D+00	6.2285D-01	4.0216D-01	3.9183D-01
4	1.1257D+00	9.6083D-01	3.4731D-01	2.7180D-01	2.6387D-01	2.5263D-01
5	3.5866D-01	1.4977D-01	1.3133D-01	1.2559D-01	9.2126D-02	8.9821D-02

TABLE 8.37

THE MAGNITUDE OF THE S ARRAY

OF MODEL FOUR

W = .3, N = 500

	1	2	3	4	4
-7	1.1011D+00	1.7636D+00	4.6087D+00	8.6322D+00	1.3133D+00
-6	1.2768D+00	1.5454D+00	2.8217D+00	3.0572D+00	2.2221D+00
-5	1.1691D+00	1.1393D+00	3.2312D+00	3.1784D+00	3.7555D+00
-4	1.1630D+00	2.2402D+01	3.2225D+00	7.5601D+00	9.7642D+00
-3	1.2914D+00	2.9842D+00	3.2131D+00	9.4326D+00	1.0795D+02
-2	1.0781D+00	2.3733D+00	3.1242D+00	8.5546D+01	1.7468D+02
-1	1.4297D+00	2.1961D+00	7.8894D+00	9.4296D+00	9.4071D+00
0	1.0386D+00	1.6825D+00	1.7977D+00	2.0551D+00	1.7593D+00
1	1.3201D+00	1.6064D+00	2.8823D+00	2.7105D+00	2.6038D+00
2	1.0919D+00	1.4397D+00	2.7366D+00	3.1998D+00	2.5026D+00
3	1.1887D+00	1.2034D+00	2.6971D+00	2.5866D+00	3.4361D+01
4	1.1822D+00	1.6062D+01	2.7056D+00	5.6044D+01	2.5035D+01
5	1.0998D+00	2.6628D+00	2.7436D+00	4.9987D+00	5.5907D+00
6	1.2746D+00	2.0767D+00	2.3279D+00	2.3960D+00	1.3214D+00
7	1.0739D+00	1.9226D+00	2.6961D+00	7.3638D+00	1.2475D+00

TABLE 8.38

THE MAGNITUDE OF THE GPAC

OF MODEL FOUR

W = .3, N = 500

	1	2	3	4	5
0	7.2644D-01	7.6612D-01	2.2786D-01	2.1794D-01	1.8702D-01
1	1.2245D+00	6.7688D-01	9.2257D-01	3.1685D-02	1.4906D-02
2	8.4552D-01	4.8245D-01	8.5170D-01	3.3923D-01	2.3183D-02
3	1.0221D+00	5.3716D-02	8.3693D-01	3.4214D-01	3.5191D+00
4	1.0111D+00	1.4099D+01	8.3731D-01	1.7633D+01	6.6663D+00
5	8.6141D-01	1.7230D+00	9.7234D-01	1.6350D+00	2.5160D+00

TABLE 8.39

THE MAGNITUDE OF THE C-TABLE

OF MODEL FOUR

W = .3, N = 500

	0	1	2	3	4	5
-8	1.0000D+00	6.2620D-01	1.8524D-01	4.6330D-03	1.3579D-04	1.1493D-05
-7	1.0000D+00	7.7515D-01	1.8155D-01	5.6774D-03	1.0929D-04	1.1521D-05
-6	1.0000D+00	6.6961D-01	1.5418D-01	1.1240D-02	3.9373D-04	1.1451D-05
-5	1.0000D+00	7.7734D-01	8.9483D-02	1.1560D-02	2.4080D-04	4.5512D-06
-4	1.0000D+00	7.6877D-01	6.3469D-03	1.3806D-02	1.3657D-05	6.8272D-07
-3	1.0000D+00	7.5213D-01	1.1816D-01	1.6495D-02	3.9916D-05	1.9400D-07
-2	1.0000D+00	8.8954D-01	2.4491D-01	1.9368D-02	1.1766D-04	8.3683D-06
-1	1.0000D+00	7.2644D-01	3.6182D-01	2.0993D-02	3.7136D-03	5.6142D-04
0	1.0000D+00	1.0000D+00	4.7228D-01	9.2132D-02	1.7040D-02	3.0018D-03
1	1.0000D+00	7.2644D-01	3.6182D-01	2.0993D-02	3.7136D-03	5.6142D-04
2	1.0000D+00	8.8954D-01	2.4491D-01	1.9368D-02	1.1766D-04	8.3683D-06
3	1.0000D+00	7.5213D-01	1.1816D-01	1.6495D-02	3.9916D-05	1.9400D-07
4	1.0000D+00	7.6877D-01	6.3469D-03	1.3806D-02	1.3657D-05	6.8272D-07
5	1.0000D+00	7.7734D-01	8.9483D-02	1.1560D-02	2.4080D-04	4.5512D-06
6	1.0000D+00	6.6961D-01	1.5418D-01	1.1240D-02	3.9373D-04	1.1451D-05
7	1.0000D+00	7.7515D-01	1.8155D-01	5.6774D-03	1.0929D-04	1.1521D-05
8	1.0000D+00	6.2620D-01	1.8524D-01	4.6330D-03	1.3579D-04	1.1493D-05

TABLE 8.40

THE SVD OF MODEL FOUR

N =500

1	5.9990D+01	5.8152D+01	5.6281D+01	5.4207D+01	5.2062D+01	4.9821D+01
2	3.8794D+00	3.8560D+00	3.8208D+00	3.6843D+00	3.5548D+00	3.2319D+00
3	3.5552D+00	3.3864D+00	2.9585D+00	2.7614D+00	2.3709D+00	2.2336D+00
4	2.8286D+00	2.2407D+00	1.9562D+00	1.3428D+00	1.0078D+00	6.5778D-01
5	7.7211D-01	7.4729D-01	5.2658D-01	4.9949D-01	3.8598D-01	2.3318D-01
6	7.1029D-01	4.8288D-01	4.6068D-01	3.2782D-01	2.1887D-01	1.6020D-01
7	3.9392D-01	3.9280D-01	2.3369D-01	1.6760D-01	1.0729D-01	5.9581D-02
8	3.9237D-01	1.8307D-01	1.4579D-01	9.0098D-02	5.3858D-02	4.8174D-02

TABLE 8.41

THE S ARRAY OF MODEL FIVE

W = .5, N = 500

	1	2	3	4	5
-7	-1.6141D+00	3.9508D+00	-4.0455D+00	4.8552D+00	3.7895D+01
-6	2.4690D+00	3.4673D+00	1.1710D+01	-8.7919D-01	-1.0795D+00
-5	-1.8598D+00	2.6948D+00	2.0734D+00	1.6219D+00	-1.0229D+00
-4	-1.5725D-01	5.0611D+00	3.5474D-01	6.6574D-01	-2.0977D+01
-3	-5.4841D+00	4.9086D+00	-4.3575D-01	4.8281D+00	-1.3049D+01
-2	-1.1363D+00	3.6244D+00	-1.1177D+01	9.9274D+00	-1.0500D+01
-1	1.3331D+01	2.7219D+00	3.6070D+00	4.5186D+00	8.7492D+00
0	-9.3022D-01	1.4132D+00	-1.0154D+00	1.3097D+00	-1.1391D+00
1	-8.3344D+00	1.7816D+00	-2.3551D+00	1.7275D+00	-3.0324D+00
2	-1.2230D+00	1.5914D+00	2.1748D-01	1.6538D-01	-6.9127D-01
3	1.8659D-01	2.2324D+00	-1.5046D-01	4.3551D-01	-5.6571D-01
4	-2.1630D+00	2.2970D+00	-6.6756D-01	5.4776D-01	-1.5688D+00
5	-7.1173D-01	1.5352D+00	-1.2480D+00	-9.1878D-01	-1.8283D+00
6	-2.6284D+00	1.6175D+00	-1.1318D+01	3.3174D+00	-3.3962D+00
7	-9.5669D-01	2.6479D-01	-5.5324D-01	3.8313D+00	-4.5119D+00

TABLE 8.42

THE GPAC OF MODEL FIVE

W = .5, N = 500

	1	2	3	4	5
0	6.9780D-02	-5.1919D-01	2.8150D-01	-2.8984D-01	1.3020D-01
1	-7.3344D+00	-4.9156D-01	-2.1071D-01	-1.7401D-01	-2.8880D-01
2	-2.2301D-01	-3.2421D-01	4.9909D-01	-3.4254D-02	-5.2973D-02
3	1.1866D+00	-4.4109D-01	4.2414D-01	-6.5416D-01	-2.6968D-02
4	-1.1630D+00	-8.5239D-01	3.2196D-01	-3.3772D-01	-1.5337D+00
5	2.8827D-01	-4.4276D-01	1.0658D-01	-1.0450D+00	-1.6937D+00

TABLE 8.43

THE C-TABLE OF MODEL FIVE

N = 500

	0	1	2	3	4	5
-8	1.0000D+00	3.5229D-02	1.3030D-03	2.1399D-04	2.3866D-04	9.4880D-05
-7	1.0000D+00	6.9528D-02	-1.9306D-03	-3.7067D-04	4.0630D-04	8.7509D-05
-6	1.0000D+00	8.2418D-02	-1.6919D-02	-3.0235D-03	5.5576D-04	-7.3779D-05
-5	1.0000D+00	-1.4564D-01	-1.9193D-02	7.0470D-04	1.3092D-03	-2.8050D-04
-4	1.0000D+00	2.4487D-02	-1.5707D-02	8.1468D-03	2.7286D-03	9.3073D-04
-3	1.0000D+00	1.0373D-01	-2.3248D-02	3.3365D-02	1.1478D-02	1.3299D-03
-2	1.0000D+00	-5.0998D-01	-2.5475D-01	1.0389D-01	6.4546D-02	-1.7532D-02
-1	1.0000D+00	5.1373D-02	-5.1262D-01	-1.6935D-01	2.0429D-01	5.2653D-02
0	1.0000D+00	1.0000D+00	-9.9736D-01	-7.3195D-01	5.0841D-01	2.9613D-01
1	1.0000D+00	5.1373D-02	-5.1262D-01	-1.6935D-01	2.0429D-01	5.2653D-02
2	1.0000D+00	-5.0998D-01	-2.5475D-01	1.0389D-01	6.4546D-02	-1.7532D-02
3	1.0000D+00	1.0373D-01	-2.3248D-02	3.3365D-02	1.1478D-02	1.3299D-03
4	1.0000D+00	2.4487D-02	-1.5707D-02	8.1468D-03	2.7286D-03	9.3073D-04
5	1.0000D+00	-1.4564D-01	-1.9193D-02	7.0470D-04	1.3092D-03	-2.8050D-04
6	1.0000D+00	8.2418D-02	-1.6919D-02	-3.0235D-03	5.5576D-04	-7.3779D-05
7	1.0000D+00	6.9528D-02	-1.9306D-03	-3.7067D-04	4.0630D-04	8.7509D-05
8	1.0000D+00	3.5229D-02	1.3030D-03	2.1399D-04	2.3866D-04	9.4880D-05

TABLE 8.44

THE SVD OF MODEL FIVE

N = 500

1	3.5909D+00	3.2491D+00	3.2245D+00	2.4982D+00	2.4932D+00	1.3596D+00
2	3.1241D+00	3.0899D+00	2.4798D+00	2.4734D+00	1.2746D+00	1.1093D+00
3	1.8170D+00	1.3665D+00	1.3048D+00	5.5563D-01	3.5323D-01	3.5323D-01
4	1.2802D+00	1.0675D+00	3.7786D-01	3.0433D-01	3.0391D-01	2.9139D-01
5	5.8339D-01	2.3379D-01	2.3233D-01	2.2454D-01	1.8040D-01	1.7866D-01

TABLE 8.45

THE S ARRAY OF MODEL ONE

W = .5, N = 200

	1	2	3	4	5
-7	-1.5310D+00	4.5527D+00	-2.4117D+03	1.3337D+01	7.3246D+00
-6	-7.7917D-01	4.0536D+00	4.7012D-01	2.1583D+00	-2.0874D+00
-5	-8.0605D+00	4.1716D+00	-1.8856D+00	2.1477D+00	4.2265D+01
-4	-2.9133D+00	4.3095D+00	-4.2821D+00	3.7226D+00	-2.0006D+01
-3	-2.4472D+00	4.3076D+00	-1.1610D+02	-8.7660D+01	1.6148D+01
-2	-2.2416D+00	4.2431D+00	8.8921D+01	9.1259D+02	-3.1453D+01
-1	-2.1062D+00	5.8060D+00	8.3723D+00	1.9904D+01	4.0641D+01
0	-1.9040D+00	2.8331D+00	-2.1168D+00	2.3687D+00	-2.2383D+00
1	-1.8054D+00	3.5848D+00	-3.3651D+00	3.3345D+00	-9.0815D+00
2	-1.6910D+00	3.5256D+00	-3.6594D+00	-2.5441D+02	4.4235D+00
3	-1.5227D+00	3.5236D+00	5.9136D+02	1.9589D+01	-3.6400D+00
4	-1.1416D+00	3.7848D+00	2.9443D+00	2.9627D+00	-2.7229D+00
5	3.5283D+00	3.5694D+00	-3.6151D-01	2.8612D+00	9.9019D+01
6	-2.8833D+00	3.4786D+00	-3.4845D+00	3.3658D+00	-2.5997D+00
7	-2.3088D+00	3.4779D+00	6.0526D+02	7.3335D+00	-5.0567D+00

TABLE 8.46

THE GPAC OF MODEL ONE

W = .5 , N = 200

	1	2	3	4	5
0	-9.0403D-01	-4.8796D-01	2.5284D-01	-1.1901D-01	5.5074D-02
1	-8.0541D-01	-8.4485D-01	3.7843D-02	-3.6539D-03	-2.8873D-01
2	-6.9097D-01	-8.1846D-01	-3.1520D-02	-2.9022D+00	-2.7393D-01
3	-5.2267D-01	-8.1763D-01	1.3810D+02	-5.2623D+00	-1.8195D-01
4	-1.4163D-01	-9.0727D-01	1.5614D+00	-1.3795D+00	6.4424D-02
5	4.5283D+00	-8.8055D-01	7.6898D-01	-1.3257D+00	4.7437D+01

TABLE 8.47

THE C-TABLE OF MODEL ONE

N = 200

	0	1	2	3	4	5
-8	1.0000D+00	-4.1569D-01	-2.3497D-02	-2.4575D-04	2.5806D-06	-6.2498D-08
-7	1.0000D+00	3.1761D-01	-3.0771D-02	-1.8379D-06	1.0159D-05	-3.6610D-07
-6	1.0000D+00	-1.6865D-01	-4.0272D-02	1.2721D-03	4.0257D-05	-1.0315D-06
-5	1.0000D+00	-3.7244D-02	-4.5735D-02	1.6543D-03	3.0368D-05	-2.1744D-08
-4	1.0000D+00	2.6296D-01	-5.0410D-02	1.0594D-03	2.2014D-05	-3.3752D-07
-3	1.0000D+00	-5.0311D-01	-6.1653D-02	7.6716D-06	4.1833D-06	1.8550D-06
-2	1.0000D+00	7.2812D-01	-7.5328D-02	-2.4339D-04	1.4414D-06	-6.7718D-06
-1	1.0000D+00	-9.0403D-01	-8.9161D-02	-6.4315D-03	3.9449D-04	2.3454D-05
0	1.0000D+00	1.0000D+00	-1.8272D-01	-2.5437D-02	3.3149D-03	4.2586D-04
1	1.0000D+00	-9.0403D-01	-8.9161D-02	-6.4315D-03	3.9449D-04	2.3454D-05
2	1.0000D+00	7.2812D-01	-7.5328D-02	-2.4339D-04	1.4414D-06	-6.7718D-06
3	1.0000D+00	-5.0311D-01	-6.1653D-02	7.6716D-06	4.1833D-06	1.8550D-06
4	1.0000D+00	2.6296D-01	-5.0410D-02	1.0594D-03	2.2014D-05	-3.3752D-07
5	1.0000D+00	-3.7244D-02	-4.5735D-02	1.6543D-03	3.0368D-05	-2.1744D-08
6	1.0000D+00	-1.6865D-01	-4.0272D-02	1.2721D-03	4.0257D-05	-1.0315D-06
7	1.0000D+00	3.1761D-01	-3.0771D-02	-1.8379D-06	1.0159D-05	-3.6610D-07
8	1.0000D+00	-4.1569D-01	-2.3497D-02	-2.4575D-04	2.5806D-06	-6.2498D-08

TABLE 8.48

THE SVD TABLE OF MODEL ONE

N = 200

1	4.5969D+01	4.3645D+01	4.0111D+01	3.5583D+01	3.0568D+01	2.5948D+01
2	1.8138D+01	1.6611D+01	1.5368D+01	1.4692D+01	1.4556D+01	1.4508D+01
3	1.3427D+00	8.4884D-01	7.1696D-01	6.3779D-01	3.0004D-01	2.8079D-01
4	5.0953D-01	4.5726D-01	3.8978D-01	1.5931D-01	1.2814D-01	1.2711D-01
5	2.8687D-01	2.5183D-01	9.8130D-02	8.2014D-02	7.2040D-02	7.2007D-02

TABLE 8.49

THE S ARRAY OF MODEL TWO

W = 0, N = 200

	1	2	3	4	5
-7	-1.4522D+00	4.1492D+00	-5.2042D+00	4.4829D+00	-5.6345D+00
-6	-4.9827D-01	4.1577D+00	-7.1127D+00	1.3725D+01	8.2844D+00
-5	-5.2894D+00	4.1585D+00	-1.0524D+01	-3.5085D+01	-2.1580D+01
-4	-2.7543D+00	4.1526D+00	3.7666D+00	1.1415D+01	-2.3940D+00
-3	-2.3723D+00	4.1747D+00	-7.0434D+00	8.4487D+00	8.6742D+01
-2	-2.1901D+00	4.1948D+00	-1.1739D+00	1.3243D+02	2.3712D+01
-1	-2.0607D+00	4.1789D+00	-1.5306D+02	1.1193D+02	7.5724D+01
0	-1.9428D+00	3.6306D+00	-3.7188D+00	3.8466D+00	-3.6607D+00
1	-1.8403D+00	3.6173D+00	1.4132D+00	9.0718D+00	-1.5858D+00
2	-1.7287D+00	3.6364D+00	-8.9914D+00	1.0442D+01	-4.1328D+00
3	-1.5700D+00	3.6621D+00	-1.7369D+00	4.5354D+00	1.7932D+00
4	-1.2331D+00	3.6519D+00	-6.0319D+00	6.7573D+00	-1.4346D+01
5	9.9311D-01	3.6485D+00	-8.7843D+00	3.7670D+01	1.1644D+01
6	-3.2112D+00	3.6495D+00	-1.8148D+01	-3.0935D+01	-2.1545D+01
7	-2.3722D+00	3.6791D+00	-4.9780D+00	4.4036D+00	-3.8363D+00

TABLE 8.50

THE GPAC OF MODEL TWO

W = 0, N = 200

	1	2	3	4	5
0	-9.4276D-01	-8.6879D-01	-2.4297D-02	-3.4365D-02	4.8342D-02
1	-8.4028D-01	-8.6233D-01	1.2038D+00	-6.8504D-02	6.6878D-02
2	-7.2872D-01	-8.7106D-01	-1.2766D+00	-1.2360D+00	4.7645D-02
3	-5.7003D-01	-8.8187D-01	4.6114D-01	-3.9732D-01	7.4907D-01
4	-2.3313D-01	-8.7817D-01	-5.7314D-01	1.9260D-01	-6.6479D-01
5	1.9931D+00	-8.7753D-01	-1.2350D+00	-2.7447D+00	-1.4056D+00

TABLE 8.51

THE C-TABLE OF MODEL TWO

N = 200

	0	1	2	3	4	5
-8	1.0000D+00	-4.6394D-01	-3.9038D-02	-4.0345D-05	-3.8465D-08	-1.1734D-10
-7	1.0000D+00	3.3811D-01	-4.3378D-02	1.2887D-04	3.4849D-07	-9.2728D-10
-6	1.0000D+00	-1.5290D-01	-4.9318D-02	-3.6956D-05	-5.0500D-08	2.4250D-10
-5	1.0000D+00	-7.6716D-02	-5.6201D-02	2.9924D-05	-1.8399D-08	-1.7253D-10
-4	1.0000D+00	3.2907D-01	-6.3998D-02	-5.2210D-05	9.5531D-08	2.5953D-10
-3	1.0000D+00	-5.7728D-01	-7.2571D-02	-1.1322D-04	2.4044D-07	3.4647D-10
-2	1.0000D+00	7.9218D-01	-8.3314D-02	8.8689D-05	1.9453D-07	7.2719D-09
-1	1.0000D+00	-9.4276D-01	-9.6615D-02	7.3676D-05	2.8397D-06	1.0873D-07
0	1.0000D+00	1.0000D+00	-1.1121D-01	-3.0323D-03	8.2635D-05	2.2492D-06
1	1.0000D+00	-9.4276D-01	-9.6615D-02	7.3676D-05	2.8397D-06	1.0873D-07
2	1.0000D+00	7.9218D-01	-8.3314D-02	8.8689D-05	1.9453D-07	7.271[illegible]D-09
3	1.0000D+00	-5.7728D-01	-7.2571D-02	-1.1322D-04	2.4044D-07	3.4647D-10
4	1.0000D+00	3.2907D-01	-6.3998D-02	-5.2210D-05	9.5531D-08	2.5953D-10
5	1.0000D+00	-7.6716D-02	-5.6201D-02	2.9924D-05	-1.8399D-08	-1.7253D-10
6	1.0000D+00	-1.5290D-01	-4.9318D-02	-3.6956D-05	-5.0500D-08	2.4250D-10
7	1.0000D+00	3.3811D-01	-4.3378D-02	1.2887D-04	3.4849D-07	-9.2728D-10
8	1.0000D+00	-4.6394D-01	-3.9038D-02	-4.0345D-05	-3.8465D-08	-1.1734D-10

TABLE 8.52

THE SVD OF MODEL TWO

N = 200

1	3.2790D+02	3.1059D+02	2.8529D+02	2.5368D+02	2.1928D+02	1.8772D+02
2	1.3568D+02	1.2854D+02	1.2345D+02	1.2092D+02	1.2051D+02	1.2008D+02
3	6.9206D+00	5.4653D+00	4.8631D+00	4.7833D+00	4.7562D+00	4.6046D+00
4	8.4793D-01	8.4792D-01	5.9610D-01	4.4816D-01	3.8749D-01	3.1849D-01
5	2.4270D-01	2.3730D-01	1.7352D-01	5.0385D-02	4.7405D-02	4.7252D-02

TABLE 8.53

THE S ARRAY OF MODEL THREE

W = .5, N = 200

	1	2	3	4	5
-7	2.0055D+01	2.1468D+00	-8.2742D+00	3.8026D+01	-1.4243D+01
-6	-1.0366D+00	2.2020D+00	-4.6432D+00	1.6140D+01	-8.2536D+01
-5	3.2416D+01	2.2577D+00	-9.2838D-02	7.0535D-01	8.5710D-01
-4	-1.0690D+00	2.3090D+00	-7.1884D-01	9.3182D-01	-4.7297D+00
-3	1.7499D+01	2.2778D+00	1.3665D+00	5.0476D+00	7.2008D+00
-2	-1.0706D+00	2.2344D+00	-1.0264D+01	1.6573D+01	-1.4403D+01
-1	1.6419D+01	2.0935D+00	4.6812D+00	8.5115D+00	3.2273D+01
0	-9.4259D-01	1.7146D+00	-1.2549D+00	1.4720D+00	-1.4078D+00
1	-1.5162D+01	1.8162D+00	-2.2815D+00	1.7790D+00	-4.3493D+00
2	-9.4594D-01	1.7848D+00	-6.7732D-01	-6.5547D-01	1.9853D-01
3	-1.5497D+01	1.8064D+00	8.3291D-01	-3.3770D-01	-5.0240D+00
4	-9.7007D-01	1.8424D+00	7.7183D-02	4.0342D+00	-1.9328D+00
5	-2.8296D+01	1.7998D+00	-3.5032D+00	3.5235D+00	-3.5538D+00
6	-9.5251D-01	1.8419D+00	-2.5098D+00	2.8589D+00	2.1464D+01
7	-1.9048D+01	1.8586D+00	-2.2368D+00	3.5843D+01	1.6306D+00

TABLE 8.54

THE GPAC OF MODEL THREE

W = .5, N = 200

	1	2	3	4	5
0	5.7407D-02	-8.1901D-01	2.6808D-01	-1.7294D-01	4.3620D-02
1	-1.4162D+01	-8.1282D-01	-2.2229D-01	-1.0734D-01	-3.0198D-01
2	5.4056D-02	-7.8357D-01	4.9567D-01	1.2986D-01	-2.7570D-02
3	-1.4497D+01	-7.8234D-01	1.1587D+00	3.6241D-01	-1.0622D+00
4	2.9926D-02	-8.1606D-01	8.3138D-01	-5.7194D+00	2.2551D+00
5	-2.7296D+01	-8.1737D-01	-7.5447D-01	-2.1831D-01	-4.3058D-02

TABLE 8.55

THE C-TABLE OF MODEL THREE

N = 200

	0	1	2	3	4	5
-8	1.0000D+00	4.4610D-01	-1.9951D-01	-3.7521D-04	3.3178D-05	1.6371D-06
-7	1.0000D+00	-2.4717D-02	-2.3277D-01	2.1298D-03	8.1687D-06	1.6613D-06
-6	1.0000D+00	-5.2042D-01	-2.7131D-01	-7.0213D-03	1.0865D-04	1.1024D-06
-5	1.0000D+00	1.9066D-02	-3.3193D-01	9.3062D-03	4.9770D-04	-2.5601D-05
-4	1.0000D+00	6.3711D-01	-4.0674D-01	1.1194D-02	8.7019D-05	-1.1353D-05
-3	1.0000D+00	-4.3948D-02	-5.1991D-01	9.6608D-03	-2.4012D-04	1.0688D-05
-2	1.0000D+00	-8.1301D-01	-6.6351D-01	1.9490D-02	1.8491D-03	-3.8765D-04
-1	1.0000D+00	5.7407D-02	-8.1631D-01	-8.7677D-02	1.7226D-02	1.2837D-03
0	1.0000D+00	1.0000D+00	-9.9670D-01	-3.2706D-01	9.9608D-02	2.9429D-02
1	1.0000D+00	5.7407D-02	-8.1631D-01	-8.7677D-02	1.7226D-02	1.2837D-03
2	1.0000D+00	-8.1301D-01	-6.6351D-01	1.9490D-02	1.8491D-03	-3.8765D-04
3	1.0000D+00	-4.3948D-02	-5.1991D-01	9.6608D-03	-2.4012D-04	1.0688D-05
4	1.0000D+00	6.3711D-01	-4.0674D-01	1.1194D-02	8.7019D-05	-1.1353D-05
5	1.0000D+00	1.9066D-02	-3.3193D-01	9.3062D-03	4.9770D-04	-2.5601D-05
6	1.0000D+00	-5.2042D-01	-2.7131D-01	-7.0213D-03	1.0865D-04	1.1024D-06
7	1.0000D+00	-2.4717D-02	-2.3277D-01	2.1298D-03	8.1687D-06	1.6613D-06
8	1.0000D+00	4.4610D-01	-1.9951D-01	-3.7521D-04	3.3178D-05	1.6371D-06

TABLE 8.56

THE SVD OF MODEL THREE

N = 200

1	1.1209D+01	1.0099D+01	1.0096D+01	8.6087D+00	8.6078D+00	6.9721D+00
2	8.8837D+00	8.8763D+00	7.6814D+00	7.6793D+00	6.2098D+00	6.2065D+00
3	1.7108D+00	1.1617D+00	1.1586D+00	7.6451D-01	6.6186D-01	6.4994D-01
4	1.0011D+00	9.4702D-01	4.2709D-01	3.2626D-01	3.2547D-01	3.1597D-01
5	3.8420D-01	2.6189D-01	2.4036D-01	1.6371D-01	1.1433D-01	1.0886D-01

TABLE 8.57

THE MAGNITUDE OF THE S ARRAY

OF MODEL FOUR

W = .3, N = 200

	1	2	3	4	5
-6	1.2288D+00	1.2349D+00	3.6708D+00	3.8737D+00	4.1256D+00
-5	1.2264D+00	7.1961D+01	3.5300D+00	4.4805D+00	3.3186D+01
-4	1.1075D+00	3.1726D+00	3.5608D+00	1.5138D+01	3.0164D+00
-3	1.4188D+00	2.5111D+00	3.1640D+00	3.1557D+00	2.0293D+01
-2	1.0291D+00	2.2679D+00	3.3892D+00	6.2669D+01	2.0152D+01
-1	1.6451D+00	2.1679D+00	9.6425D+00	1.3330D+01	1.0786D+01
0	9.9621D-01	1.6966D+00	1.7977D+00	1.9751D+00	1.7188D+00
1	1.4655D+00	1.6481D+00	2.6667D+00	3.0040D+00	2.0331D+00
2	1.0418D+00	1.5483D+00	2.9406D+00	6.7651D+00	2.0357D+00
3	1.2636D+00	1.4240D+00	2.3917D+00	2.3515D+00	1.1516D+00
4	1.1321D+00	1.1245D+00	2.4727D+00	6.1645D+00	1.1250D+01
5	1.1303D+00	5.2856D+01	2.5664D+00	1.1919D+01	5.7175D+01
6	1.2261D+00	2.5521D+00	2.5225D+00	3.6314D+00	1.6754D+00

TABLE 8.58

THE MAGNITUDE OF THE GPAC

OF MODEL FOUR

W = .3, N = 200

	1	2	3	4	5
0	6.0556D-01	7.8261D-01	1.8643D-01	1.4817D-01	1.5936D-01
1	1.4240D+00	7.2671D-01	7.8682D-01	4.7935D-02	1.0089D-01
2	7.3427D-01	6.1658D-01	9.2940D-01	2.1438D+00	1.0031D-01
3	1.1410D+00	4.4885D-01	6.7168D-01	1.5534D-01	3.8179D-01
4	9.2310D-01	1.5627D-02	7.0047D-01	1.3759D+00	3.3898D-01
5	9.1987D-01	4.2803D+01	6.9915D-01	3.0770D+00	1.3859D+01

TABLE 8.59

THE MAGNITUDE OF THE C-TABLE

OF MODEL FOUR

W = .3, N = 200

	0	1	2	3	4	5
-8	1.0000D+00	5.4192D-01	1.5365D-01	5.5308D-03	4.5881D-05	2.2008D-05
-7	1.0000D+00	6.6429D-01	1.0883D-01	2.4074D-03	3.0097D-04	3.0601D-05
-6	1.0000D+00	6.1347D-01	6.6674D-02	6.9700D-03	3.6868D-04	2.4675D-05
-5	1.0000D+00	6.6691D-01	1.5577D-03	9.9693D-03	1.1982D-04	1.7805D-06
-4	1.0000D+00	7.2247D-01	9.9682D-02	1.4232D-02	8.7084D-05	5.2523D-06
-3	1.0000D+00	6.3319D-01	2.2208D-01	2.1189D-02	5.6060D-04	1.3757D-05
-2	1.0000D+00	8.6233D-01	3.6018D-01	2.2799D-02	2.6150D-04	1.3714D-04
-1	1.0000D+00	6.0556D-01	4.9563D-01	2.8976D-02	5.4554D-03	1.3594D-03
0	1.0000D+00	1.0000D+00	6.3330D-01	1.5542D-01	3.6817D-02	8.5300D-03
1	1.0000D+00	6.0556D-01	4.9563D-01	2.8976D-02	5.4554D-03	1.3594D-03
2	1.0000D+00	8.6233D-01	3.6018D-01	2.2799D-02	2.6150D-04	1.3714D-04
3	1.0000D+00	6.3319D-01	2.2208D-01	2.1189D-02	5,6060D-04	1.3757D-05
4	1.0000D+00	7.2247D-01	9.9682D-02	1.4232D-02	8.7084D-05	5.2523D-06
5	1.0000D+00	6.6691D-01	1.5577D-03	9.9693D-03	1.1982D-04	1.7805D-06
6	1.0000D+00	6.1347D-01	6.6674D-02	6.9700D-03	3.6868D-04	2.4675D-05
7	1.0000D+00	6.6429D-01	1.0883D-01	2.4074D-03	3.0097D-04	3.0601D-05
8	1.0000D+00	5.4192D-01	1.5365D-01	5.5308D-03	4.5881D-05	2.2008D-05

TABLE 8.60

THE SVD OF MODEL FOUR

N = 200

1	4.5633D+01	4.4197D+01	4.2738D+01	4.1093D+01	3.9391D+01	3.7611D+01
2	5.3943D+00	5.2934D+00	5.1754D+00	4.8961D+00	4.6174D+00	4.1233D+00
3	3.6367D+00	3.4278D+00	3.0136D+00	2.7967D+00	2.4327D+00	2.3419D+00
4	2.6065D+00	2.0662D+00	1.8050D+00	1.2495D+00	9.4865D-01	7.3255D-01
5	8.8711D-01	8.5820D-01	5.9385D-01	5.8481D-01	5.4729D-01	2.9298D-01
6	8.3765D-01	5.8437D-01	5.7776D-01	4.9842D-01	2.7188D-01	2.0098D-01
7	4.6006D-01	4.5841D-01	3.6366D-01	1.9176D-01	1.5701D-01	1.5660D-01
8	4.5636D-01	2.8595D-01	1.7588D-01	1.5469D-01	1.5448D-01	1.3596D-01

TABLE 8.61

THE S ARRAY OF MODEL FIVE

W = .5, N = 200

	1	2	3	4	5
-7	2.9012D+00	-9.2606D+00	-5.9701D+00	3.9105D+00	-2.0782D+00
-6	6.1954D-01	1.6713D+00	-2.5590D+00	-1.1695D+01	-2.6069D+00
-5	-4.9291D-01	1.9547D+00	-1.7894D+01	-1.4487D+00	-6.5365D+00
-4	-5.6692D+00	2.8660D+00	-3.6790D+00	5.0234D+00	-5.8570D+00
-3	6.3538D-01	2.0995D+00	8.6135D-02	-5.1676D+00	-1.4950D+01
-2	-1.2244D+00	1.9637D+00	5.0816D+00	-5.8594D+00	-1.4168D+01
-1	8.4206D+00	2.7184D+00	-7.4911D+00	1.1658D+01	-1.3476D+01
0	-8.9385D-01	1.3317D+00	-1.6197D+00	1.8810D+00	-2.1862D+00
1	-5.4570D+00	1.0319D+00	-6.7229D-01	2.9506D-01	-1.6654D+00
2	-3.8852D-01	9.3209D-01	-3.9167D-02	2.3541D+00	-1.6913D+00
3	-1.2142D+00	9.9384D-01	-2.3149D+00	2.2759D+00	-1.2091D+00
4	9.7205D-01	5.1153D-01	-6.0909D-01	1.7209D-01	-7.4338D-01
5	-3.8254D-01	5.4255D-01	-2.2976D+00	9.0814D-01	-5.8374D-01
6	-7.4367D-01	7.9699D-01	-1.1068D+00	3.2026D+00	-4.8321D-01
7	-5.4730D-01	2.8237D+00	1.6868D+00	-4.2604D-01	-1.9395D+00

TABLE 8.62

THE GPAC OF MODEL FIVE

W = .5, N = 200

	1	2	3	4	5
0	1.0615D-01	-4.8990D-01	-2.1622D-01	-1.6136D-01	-1.6222D-01
1	-4.4570D+00	-5.2551D-01	1.3230D-01	5.0357D-02	-1.1755D-01
2	6.1148D-01	-4.4396D-01	4.5472D-01	4.5556D-01	-1.1313D-01
3	-2.1417D-01	-3.4677D-01	-6.2923D-01	-4.5306D-01	-2.0644D-01
4	1.9721D+00	-2.6169D-01	-3.4039D-02	1.1878D-01	-1.1373D-01
5	6.1746D-01	-3.2463D-01	-8.9784D-01	7.7649D-02	-2.2392D-01

TABLE 8.63

THE C-TABLE OF MODEL FIVE

N = 200

	0	1	2	3	4	5
-8	1.0000D+00	8.7548D-03	-2.2481D-04	-1.6781D-05	1.4237D-06	5.9225D-07
-7	1.0000D+00	1.9339D-02	2.8651D-04	3.4454D-05	6.7418D-06	-9.7904D-07
-6	1.0000D+00	7.5446D-02	-3.3291D-03	-1.8584D-04	8.2321D-06	4.2107D-06
-5	1.0000D+00	1.2219D-01	-1.0255D-02	2.0699D-04	-1.0602D-04	-1.8804D-05
-4	1.0000D+00	6.1959D-02	-3.9188D-02	-6.0808D-03	8.9251D-04	1.6535D-04
-3	1.0000D+00	-2.8930D-01	-1.1301D-01	9.6639D-03	1.9700D-03	-8.0095D-04
-2	1.0000D+00	-4.7311D-01	-2.5455D-01	2.1253D-02	-4.3243D-03	7.0799D-03
-1	1.0000D+00	1.0615D-01	-4.8438D-01	1.6064D-01	8.5873D-02	-6.0231D-02
0	1.0000D+00	1.0000D+00	-9.8873D-01	-7.4296D-01	5.3219D-01	3.7128D-01
1	1.0000D+00	1.0615D-01	-4.8438D-01	1.6064D-01	8.5873D-02	-6.0231D-02
2	1.0000D+00	-4.7311D-01	-2.5455D-01	2.1253D-02	-4.3243D-03	7.0799D-03
3	1.0000D+00	-2.8930D-01	-1.1301D-01	9.6639D-03	1.9700D-03	-8.0095D-04
4	1.0000D+00	6.1959D-02	-3.9188D-02	-6.0808D-03	8.9251D-04	1.6535D-04
5	1.0000D+00	1.2219D-01	-1.0255D-02	2.0699D-04	-1.0602D-04	-1.8804D-05
6	1.0000D+00	7.5446D-02	-3.3291D-03	-1.8584D-04	8.2321D-06	4.2107D-06
7	1.0000D+00	1.9339D-02	2.8651D-04	3.4454D-05	6.7418D-06	-9.7904D-07
8	1.0000D+00	8.7548D-03	-2.2481D-04	-1.6781D-05	1.4237D-06	5.9225D-07

TABLE 8.64

THE SVD OF MODEL FIVE

N = 200

1	2.4746D+00	2.4461D+00	2.2399D+00	1.9436D+00	1.7845D+00	9.5233D-01
2	2.4329D+00	2.0998D+00	1.8750D+00	1.6127D+00	9.1588D-01	8.3405D-01
3	1.0837D+00	8.6520D-01	7.8935D-01	3.5624D-01	2.0267D-01	2.0067D-01
4	7.7736D-01	7.1960D-01	2.8829D-01	1.9448D-01	1.9416D-01	1.7157D-01
5	5.4994D-01	1.8714D-01	1.0901D-01	8.2445D-02	6.9250D-02	6.9208D-02

TABLE 8.65

THE S ARRAY OF MODEL ONE

W = 0, N = 200

	1	2	3	4	5
-7	-1.3[illegible]D+00	1.8908D-01	-2.2281D-01	-5.[illegible]D-01	-1.0253D-01
-6	5.3[illegible]D+00	1.9665D-01	-1.6084D+00	-1.2503D-01	-1.4[illegible]D-01
-5	[illegible]D-01	1.4210D-01	-1.0681D-01	6.5822D-02	-4.1770D-02
-4	5.2333D-01	1.3354D-01	4.7266D-02	-2.9852D-02	-2.7021D+00
-3	3.2[illegible]D-01	1.2437D-01	2.8712D-02	1.0748D+01	-3.7837D+00
-2	1.9018D-01	1.1769D-01	-3.4107D+00	4.0465D+00	-4.6790D+00
-1	[illegible]-02	2.3467D-01	-6.7905D-01	1.9491D+00	-8.4270D+00
0	-7.[illegible]-02	1.1735D-01	-1.4157D-01	1.5301D-01	-1.5594D-01
1	-1.5[illegible]-01	9.2841D-02	-9.9701D-02	1.1979D-01	-2.0532D-01
2	-2.[illegible]-01	9.6260D-02	-1.8579D-02	2.7654D-02	3.9447[illegible]-01
3	-3.[illegible]-01	1.0098D-01	-2.7726D-02	7.2645D-01	2.439[illegible]
4	-4.[illegible]-01	1.0607D-01	4.2378D-01	-2.4[illegible]-01	1.43[illegible]-01
5	-[illegible].4277[illegible]-01	1.4002D-01	-1.5359D-01	8.2533D-02	-1.4846D-0[illegible]
6	-3.[illegible]D+00	1.3451D-01	-1.1484D+00	2.7312D-01	-1.2305D-01
7	[illegible]-01	1.7335D-01	-3.9436D-01	7.2540D-01	-2.6561D-01

TABLE 8.66

THE S ARRAY OF MODEL ONE

W = .5, N = 200

	1	2	3	4	5
-7	-[illegible].6177D-01	4.6159D+00	-3.9736D+00	7.7828D+00	-4.5907D+00
-6	-7.[illegible]D+00	4.6122D+00	2.9652D+01	1.0618D+00	-5.3098D+00
-5	-2.[illegible]D+00	4.5373D+00	-5.3650D+00	4.6042D+00	-4.400[illegible]
-4	-[illegible].5233D+00	4.5114D+00	-1.0285D+01	9.4099D+00	1.8169D[illegible]
-3	-2.3208D+00	4.4495D+00	-9.6578D+00	2.3546D+03	-1.5925[illegible]
-2	-2.1902D+00	4.417[illegible]+00	1.1345D+02	1.0117D+02	3.6684[illegible]1
-1	-[illegible].0649D+00	5.7649D+00	1.0915D+01	3.1331D+01	1.3054D+02
0	-1.4218D+00	2.8829D+00	-2.2805D+00	2.4595D+00	-2.4140D+00
1	-1.6402D+00	3.4049D+00	-3.3164D+00	2.9934D+00	-1.6185D+00
2	-1.7571D+00	3.4455D+00	-6.3501D+00	6.0582D+00	-1.6612D+01
3	-[illegible]D+00	3.4114D+00	-6.0334D+00	-2.2899D+02	1.4404D+01
4	-1.5[illegible]3D+00	3.3768D+00	-2.1286D+01	-1.7254D+01	-1.5153[illegible]+01
5	-[illegible]D+00	3.2[illegible]+00	-2.7503D+00	-7.0344D-01	5.2939D+00
6	1.[illegible]+00	3.29[illegible]D+00	2.0480D+01	-4.2352D+00	5.5096D+00
7	-2.[illegible]+00	3.3105D+00	2.0729D+00	3.8312D+00	3.2990D+00

CHAPTER IX

CONCLUSIONS

In this work four methods for order determination of ARMA(p,q) processes using time series data were discussed. First the GPAC, then the S and R arrays were described, and the relationship between the GPAC and the S and R arrays was stated. Model five of chapter eight was a clear example of how the S array can be more decisive in determining the order than the GPAC. This due to the fact that the GPAC is a ratio of S array entries (some information will be lost in case of having ratios instead of the exact values). Then the C-table method was discussed and the relationship between the C-table and the S and R arrays was mentioned. The C-table is recommended to be a supporting method of order determination, where the GPAC and the S and R arrays are the major ones. This due to the fact that looking for zero squares is not an easy job especially with noisy data. It is also less accurate than the S array.

The last method is the SVD, in which linear algebra was used to find the number of nonzero singular values of the matrix A which contains the autocorrelation function of an ARMA(p,q) process. The number of nonzero singular values = p = rank(A). The method proposed worked very well using the exact autocorrelation function, but it was poor for the estimated autocorrelation function.

The important point in this work was using $e^{j2\pi wm}f_m$ instead of f_m. In their work, Gray, Kelly, and McIntire (1978) used f_m and $(-1)^m f_m$, where the first corresponds to w = 0 and the second corresponds to w = .5, and they showed that the S array is clearer using f_m for high frequency data and $(-1)^m f_m$ for low frequency data. In this work we were looking for the value of w that will maximize the magnitude of C3, which will lead to a clearer pattern. We did not worry too much about C4, since the ratio of the magnitude of C4 to the magnitude of C3 is constant.

C3 is the reciprocal of the frequency response of the AR part of the ARMA(p,q) process. The w at which the frequency response is minimum will be chosen to calculate the S array; this will guarantee maximum C3 and a better pattern.

For future studies, it would be useful to modify the theorems to use the impulse response instead of the autocorrelation function. Although the impulse response satisfies equation [14], it does not have the symmetry around zero which was a useful property of the autocorrelation function. It would also be interesting to apply the concept of $e^{j2\pi wm}f_m$ to a wider variety of experimental data.

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